From Newton to Einstein:
A guided tour through space and time

with Carla Cederbaum
Outline of our tour

1. Sir Isaac Newton
   1643-1727
Why are the planets orbiting the sun?
Why are the planets orbiting the sun?
Why are the planets orbiting the sun?
Why are the planets orbiting the sun?
Newton’s new math

- rate of change/derivative

\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]

- vectors:
velocity, acceleration, force
Newton’s law of gravity

\[ \vec{F} = -\frac{mM G \vec{r}}{r^3} \]

- \( m \) = mass of planet
- \( M \) = mass of sun
- \( G \) = gravitational constant
- \( \vec{r} \) = distance planet to sun
How do we measure mass?
Outline of our tour

1. Siméon Denis Poisson 1781-1840
2. Sir Isaac Newton 1643-1727
3. Pierre Simon Laplace 1749-1827
4. Betelgeuse in Orion

Copyright © 2011 Greg Parker and Noël Carboni
Transform Newton’s ideas into math!

Modeling gravitation with mathematics (vector calculus) allows to compute predictions and improve understanding!
Vector calculus

Idea: generalize calculus to 3-dimensional space!

\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]

\[ \frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{x \to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0} \]
Newton’s idea revisited

\[ U = \text{Newtonian potential of sun} \]
\[ G = \text{gravitational constant} \]
\[ \rho = \text{mass density} = \text{mass/volume} \]
\[ \Delta = \text{“differential operator”} \]

\[ \Delta U = 4\pi G \rho \]
Where is $\vec{F}$?

$U = \text{Newtonian potential of sun}$

$m = \text{mass of planet}$

$\vec{\nabla} = \text{a differential operator}$

\[
\vec{F} = -m \vec{\nabla} U
\]
What is now mass $M$?

$M = \text{mass of sun}$

$\vec{n} = \text{normal vector to surface}$
What is now mass $M$?

Apply mathematical theorems (by Gauß and Stokes)

$$M = \iiint_{\text{sun}} \rho \, dV$$

$$= \ldots$$

$$= \frac{1}{4\pi G} \iint_{\text{surface of sun}} \vec{\nabla}U \cdot \vec{n} \, dS$$
Summary

New math allows to
- write Newton’s ideas as “differential equation”\[ \Delta U = 4\pi G \rho \]
- express mass as an integral (using mathematical theorems)

\[
M = \frac{1}{4\pi G} \iiint_{\text{surface of sun}} \vec{\nabla} U \cdot \vec{n} \, dS
\]
Bottom Line

• Use new math to “model” gravitation mathematically.
  – gives better methods for predictions
  – helps understand gravity better
• Newton’s new physics inspired new math!
Outline of our tour

1. Sir Isaac Newton
   1643-1727

2. Pierre Simon Laplace
   1749-1827

3. Siméon Denis Poisson
   1781-1840

4. Carl Friedrich Gauß
   1777-1855

5. Bernhard Riemann
   1826-1866
How can we measure curvature?
How can we measure curvature?

\[ \alpha + \beta + \gamma = 180^\circ \]

\[ \alpha + \beta + \gamma \neq 180^\circ \]
Curvature is important for:

- Geodesy and Geography
- Astronomy
- Physics
- Engineering (wings of planes,...)
- Biology (surface of cells,...)
- Mathematics

\[ \rightarrow \text{differential geometry} \]
Differential Geometry
- studies curves and surfaces
- generalizes vector calculus
- allows rigorous definition of curvature
  (in terms of derivatives)
Curvature

- Curves can be curved.
- Surfaces can be curved.
- 3-dimensional space can also be curved!
- Can even think about higher dimensional (curved) space!!
Outline of our tour

1. Sir Isaac Newton 1643-1727
2. Pierre Simon Laplace 1749-1827
3. Bernhard Riemann 1826-1866
4. Albert Einstein 1879-1955
5. Siméon Denis Poisson 1781-1840
6. Carl Friedrich Gauß 1777-1855

Betelgeuse in Orion
Why are the planets orbiting the sun?

Conflicts with observations and electrodynamics!
General Relativity

gravitation = curvature

Modeling gravitation with mathematics (differential geometry) allows to compute predictions and improve understanding!
Math allows to make predictions like:

- Black holes:

- Expansion of universe:

- Gravitational waves?
Einstein‘s theory

- is called “general relativity”
- uses ideas from differential geometry like curvature
- describes gravitational effects by a differential equation
General relativity

Main equation in “space-time”:

\[ \text{Ric} - \frac{1}{2} g R = \frac{8\pi G}{c^4} T \]

c = speed of light
R, Ric: measure curvature
g: measures distance/angles
T: describes matter
Einstein’s theory is consistent with many measurements:
- bending of light
- gravitational red shift
- ...
Applications

- **General Positioning System**
- satellites
- space travel
General relativity in every day life:
General relativity in every day life: matter curves space-time
General relativity in every day life:
General relativity in every day life: curvature influences movement
General relativity in every day life:
Bottom Line

• Again: Use math to model gravitation.
  – gives better methods for predictions
  – helps to better understand gravity
• Gauß/Riemann’s new math allows to predict new physics!
Outline of our tour

1. Sir Isaac Newton 1643-1727
2. Pierre Simon Laplace 1749-1827
3. Carl Friedrich Gauss 1777-1855
4. Bernhard Riemann 1826-1866
5. Albert Einstein 1879-1955

today

Betelgeuse in Orion
Can we forget about Newton?

Naive Idea: Yes!

Einstein’s general relativity is much better
(in predicting observations)
And much more beautiful!

But: also more difficult and less intuitive!
Can we forget about Newton?

Better: Reconcile the theories:
Think of Newton‘s theory
as an approximation to Einstein‘s!

Also:
Try to learn from Newton‘s theory how to
interpret relativistic notions!
Example: What is mass in general relativity?

Negative mass?

Many different definitions

Hawking

At infinity?

ADM

Copyright © 2011 Greg Parker and Noël Carboni
What is a good *local* definition of relativistic mass?

Step 1: differential geometry
+ Newtonian gravity
= new formula for mass

Step 2: use Newtonian limit by → to compare new definition with Newtonian mass
Mass in general relativity

new formula for mass
(analogy to Newtonian formula):

$$M = \frac{1}{4\pi G} \int \int \nabla U \cdot \vec{n} \, dS$$

constructed from geometry of static space-time
Theorem [C. ‘11]

Let

\[ m_{ADM}(E^n, g) := \frac{c^2}{16\pi G} \lim_{r \to \infty} \int_{S^{n-1}} \sum_{i=1}^{3} (g_{ii,i} - g_{ij,i}) \nu^j \, d\sigma \]

and

\[ m_{PN}(\Sigma) := \frac{1}{4\pi G} \int_{\Sigma} \frac{\partial U}{\partial \nu} \, d\sigma \]

on every okay surface in a static space-time.

Then

\[ M = m_{ADM}(E^3, g) = m_{PN}(\Sigma) \]
Step 2: Newtonian limit

Newton’s theory: $c = \infty$
Einstein’s theory: $c = 300.000 \text{km/s}$

Newtonian limit:
\[ \text{take } c \text{ to infinity} \]
Theorem 6.4.1 (Newtonian Limit of Mass Theorem). Let $\mathcal{F}(\lambda) := (\mathbb{R} \times \mathbb{E}^3, s^\alpha{}^\beta(\lambda), t_\alpha{}^\beta(\lambda), \Gamma^\nu{}_{\alpha\beta}(\lambda), T^\alpha{}_{\beta}(\lambda), \lambda)$ be a family of static isolated ends in frame theory parametrized by $\lambda \in (0, \varepsilon)$ for some $\varepsilon > 0$ and let $\mathcal{F}(0) := (\mathbb{R} \times \mathbb{E}^3, s^\alpha{}^\beta(0), t_\alpha{}^\beta(0), \Gamma^\nu{}_{\alpha\beta}(0), T^\alpha{}_{\beta}(0), 0)$ be a static isolated system of IIT with global Cartesian coordinates $(x^k(0))$. Assume that there exist global asymptotically flat systems of coordinates $(x^k(\lambda))$ for $\mathcal{F}(\lambda)$ converging to $(x^k(0))$ uniformly on $M^3$ as $\lambda \to 0$. Let $\gamma_{ij}(\lambda), \gamma_{ij}(\lambda), \gamma_{ij}(0), U(\lambda)$, and $U(0)$ denote the physical and pseudo-Newtonian metrics and potentials of $\mathcal{F}(\lambda)$ and $\mathcal{F}(0)$, respectively. Then

$$m_{\text{ADM}}(g^3(\lambda)) = m_{\text{PNFT}}(\gamma(\lambda), U(\lambda)) \to m_{\text{PNFT}}(\gamma(0), U(0)) = m_N(U(0))$$

as $\lambda \to 0$. 


When is relativistic mass approximatively Newtonian mass?

Result: If a star or black hole does not move 0 m/h then its relativistic mass is approximately equal to its Newtonian mass.
How do we find center of mass?

Newton:
What is the center of mass in general relativity?

Many different definitions

Huisken-Yau
Metzger
ADM

All at infinity

Huang
What is a good *local* definition of relativistic center of mass?

Step 1: differential geometry
+ Newtonian gravity
= new formula for center of mass

Step 2: use Newtonian limit by → to compare new definition with Newtonian center of mass
CoM in general relativity

new formula for center of mass
(analogy to Newtonian formula):

\[ \bar{z} = \frac{1}{4\pi GM} \int \int_{\text{surface of sun}} \left( \tilde{\nabla} U \cdot \bar{n} \bar{x} - \tilde{\nabla} \bar{x} \cdot \bar{n} U \right) dS \]

\( U, \bar{n}, dS, \tilde{\nabla} \) constructed from geometry of static space-time
Theorem [C. ‘11]

Let

\[ z_{ADM}^k(E^3, g) := \frac{c^2}{16\pi m G} \lim_{r \to \infty} \int_{S^2} \left[ \sum_{i=1}^{3} x^k (g_{ij,i} - g_{ii,j}) \nu^j - \sum_{i=1}^{3} (g_i^k \nu^j - g_i^j \nu^k) \right] d\sigma \]

and

\[ \tilde{z} = \frac{1}{4\pi GM} \int \int_{\text{surface of sun}} \left( \tilde{\nabla} U \cdot \tilde{n} \tilde{x} - \tilde{\nabla} \tilde{x} \cdot \tilde{n} U \right) dS \]

on every okay surface in a static space-time.

Then

\[ \tilde{z} = z_{ADM}^k(E^3, g) \]
Theorem 6.4.2 (Newtonian Limit of Center of Mass Theorem). Let $k \in \mathbb{N}$, $k \geq 3$, $\tau > 1/2$ such that $-\tau$ is non-exceptional. Let $\mathcal{F}(\lambda) := (\mathbb{R} \times E^3, s^{a\beta}(\lambda), t_{\alpha\beta}(\lambda), \Gamma_{\alpha\beta}^\mu(\lambda), T^{\alpha\beta}(\lambda), \lambda)$ be a family of $(k, \tau)$-static isolated ends in frame theory parametrized by $\lambda \in (0, \varepsilon)$ for some $\varepsilon > 0$ and let $\mathcal{F}(0) := (\mathbb{R} \times E^3, s^{a\beta}(0), t_{\alpha\beta}(0), \Gamma_{\alpha\beta}^\mu(0), T^{\alpha\beta}(0), 0)$ be a $(k, \tau)$-static isolated system of FT with global Cartesian coordinates $(x^k(0))$. Assume that there exist global wave harmonic $(k, \tau)$-asymptotically flat systems of coordinates $(x^k(\lambda))$ for $\mathcal{F}(\lambda)$ converging to $(x^k(0))$ uniformly on $M^3$ as $\lambda \to 0$. Let $g_{ij}(\lambda), N(\lambda), \gamma_{ij}(\lambda), \gamma_{ij}(0), U(\lambda)$, and $U(0)$ denote the physical and pseudo-Newtonian metrics and potentials of $\mathcal{F}(\lambda)$ and $\mathcal{F}(0)$, respectively. Finally, assume that $m_{PNFT}(\gamma(\lambda), U(\lambda))$ and $m_{PNFT}(\gamma(0), U(0))$ are non-vanishing. Then

$$
\tilde{\mathcal{Z}}_{ADM}(\gamma(\lambda)) = \tilde{\mathcal{Z}}_{A}(\gamma(\lambda), N(\lambda)) = \tilde{\mathcal{Z}}_{CMC}(\gamma(\lambda)) = \tilde{\mathcal{Z}}_{I}(\gamma(\lambda)) = \tilde{\mathcal{Z}}_{PNFT}(\gamma(\lambda), U(\lambda))
$$

$$
\to \tilde{\mathcal{Z}}_{PNFT}(\gamma(0), U(0)) = \tilde{\mathcal{Z}}_{N}(U(0)) \in \mathbb{R}^3
$$

as $\lambda \to 0$. 
When is relativistic center of mass approximatively Newtonian center?

Result: If a star or black hole does not move

\[ 0 \text{ m/h} \]

then its relativistic center of mass is approximately equal to its Newtonian center of mass.
Step 1: get new formulas of mass and center of mass inspired by Newtonian formulas in differential geometric language

Step 2: use Newtonian limit to compare new definitions with Newtonian concepts: approximate!
What is the Newtonian Limit?

See movie
Credit for pictures:

- www.wikipedia.org
- www.myflyprofile.com
- www2.ed.gov
- www.universe-review.ca
- www.newscientist.com
- www.flickr.com/photos/ak42/2971239293
- www.beyonddieting.com
- www.ugr.es