LECTURE NOTES

1. FIRST AND SECOND VARIATION OF AREA AND ISOPERIMETRY

Let \((M, g)\) be a complete Riemannian manifold without boundary and of dimension \(n \geq 2\).

Let \(\Sigma \subset M\) be a submanifold.

Let \(I \subset \mathbb{R}\) be an open interval containing the origin. Let \(\Phi : \Sigma \times I \to M\) be a smooth map such that for every \(t \in I\) the function \(\phi_t : \Sigma \to M\) given by \(\sigma \mapsto \Phi(\sigma, t)\) is an embedding and such that \(\phi_0 : \Sigma \to M\) is the inclusion. Let \(\Sigma_t = \phi_t(\Sigma)\). We say that \(\{\Sigma_t\}_{t \in I}\) is a variation of \(\Sigma\) parametrized by \(\Phi : \Sigma \times I \to M\). There are many ways to parametrize a given variation \(\{\Sigma_t\}_{t \in I}\) of \(\Sigma\).

We specialize to the case where \(\Sigma \subset M\) is a closed hypersurface with trivial normal bundle. Let \(\nu\) be a unit normal field.

A given variation \(\{\Sigma_t\}_{t \in I}\) of \(\Sigma\) can be parametrized by a function \(\Phi : \Sigma \times I \to M\) of the special form \((\sigma, t) \mapsto \exp F(\sigma, t) \nu(\sigma)\) where \(F : \Sigma \times I \to \mathbb{R}\) is a smooth function, possibly after replacing \(I\) be a smaller interval. Such parametrizations of a given variation are called normal.

The formulae for the first and second variation of area are as follows:

\[
\left.\frac{d}{dt}\right|_{t=0} \text{area}(\Sigma_t) = \int_{\Sigma} HF_t,
\]

\[
\left.\frac{d^2}{dt^2}\right|_{t=0} \text{area}(\Sigma_t) = \int_{\Sigma} H^2 F_t^2 + HF_{tt} + |\nabla F_t|^2 - (\text{Rc}(\nu, \nu) + |h|^2) F_t^2,
\]

where \(F_t\) and \(F_{tt}\) are evaluated at \(t = 0\). Note that \(F_t(\sigma, 0) = g(\nu, X)(\sigma)\) where \(X(\sigma) = (\Phi_* \circ \partial_t)(\sigma, 0)\) for all \(\sigma \in \Sigma\).

It follows that \(\Sigma\) is a critical point for area if and only if \(H = 0\). It is a stable critical point if in addition

\[
\int_{\Sigma} |\nabla f|^2 \geq \int_{\Sigma} (\text{Rc}(\nu, \nu) + |h|^2) f^2 \text{ for all } f \in C^\infty(\Sigma).
\]

We can measure the volume of \(\Sigma_t\) relative to \(\Sigma\). We have that

\[
\left.\frac{d}{dt}\right|_{t=0} \text{rel vol}(\Sigma_t) = \int_{\Sigma} F_t,
\]

\[
\left.\frac{d^2}{dt^2}\right|_{t=0} \text{rel vol}(\Sigma_t) = \int_{\Sigma} (HF_t^2 + F_{tt}).
\]

We see that \(\Sigma\) is a critical point for area with a volume constraint (i.e. the isoperimetric problem) if and only if the mean curvature \(H\) is constant. It is a stable critical point if in addition

\[
\int_{\Sigma} |\nabla f|^2 \geq \int_{\Sigma} (\text{Rc}(\nu, \nu) + |h|^2) f^2 \text{ for all } f \in C^\infty(\Sigma) \text{ with } \int_{\Sigma} f = 0.
\]

If \(\Omega\) is a smooth bounded region in \(\mathbb{R}^m\) then

\[
|\Omega| \leq c_n |\partial \Omega|^{n/(n-1)}
\]

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where the constant $c_n$ is chosen such that equality holds when $\Omega$ is a coordinate ball. Equality holds if and only if $\Omega$ is a coordinate ball. This is the (sharp) isoperimetric inequality in Euclidean space. The first completely rigorous proof was given by E. Schmidt in a series of papers in the 1940’s.

Can you give an elementary proof when $n = 2$?

Around the same time, A. Alexandrov characterized the critical surfaces for the isoperimetric problem, i.e. closed constant mean curvature embedded surfaces. They are again spheres. H. Hopf showed that the only constant mean curvature spheres that are immersed into $\mathbb{R}^3$ are round spheres. J. Barbosa and M. do Carmo showed that the only closed immersed stable constant mean curvature surfaces in $\mathbb{R}^n$ are round spheres. To everyone’s surprise, there are examples of higher genus immersed constant mean curvature surfaces in Euclidean space. In the 1980’s, examples were given by H. Wente using integrable system methods and by N. Kapouleas using gluing techniques.

E. Schmitt also establishes (sharp) isoperimetric inequalities in hyperbolic space and the sphere. There, coordinate balls are replaced by geodesic balls.

It is a natural question to ask what information about a manifold is contained in its isoperimetric surfaces. Unlike curvature, isoperimetric surfaces are objects of “global” geometry. They react sensitively to slight changes in local geometry. Changing the Euclidean metric on $\mathbb{R}^n$ in the neighborhood of any point could change every isoperimetric surface. I find that amazing. On the other hand, isoperimetric surfaces are computed from $C^0$ information on the metric. (By contrast, curvature needs two derivatives of the metric.) From that point of view, isoperimetry is a more stable concept than geometry. In these lectures, we will explore some of these connections.

Given $V \in (0, \text{vol}(M))$, we let

$$A(V) := \inf \{|\partial \Omega| : \Omega \subset M \text{ smooth and bounded with } \text{vol}(\Omega) = V \}.$$ 

It is called the isoperimetric profile function. A smooth region $\Omega \subset M$ of volume $V$ and boundary area $A(V)$ is called an isoperimetric region. Boundaries of isoperimetric regions are called isoperimetric surfaces. When $M$ is closed and if $m < 8$, then isoperimetric regions exist for every volume. This is a deep result from geometric measure theory. It will be assumed in these lectures. We remark that M. Ritoré has given examples of complete non-compact rotationally symmetric Riemannian surfaces that do not contain any isoperimetric regions.

2. Asymptotically flat manifolds

Let $M$ be a 3-dimensional manifold. We say that a Riemannian metric $g$ on $M$ is asymptotically flat if there is a compact subset $K$ of $M$ whose complement consists of finitely many components each of which is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and such that the components of the metric in these coordinate systems satisfy

$$g_{ij} = \delta_{ij} + O(|x|^{-1}) \text{ as } |x| \to \infty$$

with appropriate decay of higher order derivatives. Each component is called an end of $(M, g)$. If the boundary of $(M, g)$ is non-empty, we assume that its components are minimal surfaces.

The open coordinate ball and the coordinate sphere of radius $r$ and center at the origin with respect to the asymptotically flat coordinate system are denoted by $B_r$ and $S_r$ respectively.

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Note that asymptotically flat manifolds are complete. $\mathbb{R}^3$ with the standard Euclidean metric is asymptotically flat.

**Example 2.1.** Let $m > 0$. Then

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 \sum_{i=1}^{3} dx^i \otimes dx^i$$

is an asymptotically flat metric on $\mathbb{R}^3 \setminus \{0\}$. The asymptotically flat manifold $(\mathbb{R}^3 \setminus \{0\}, g_m)$ has two ends. The map $x \mapsto \frac{m^2}{4|x|^2}$ is a $\mathbb{Z}_2$-isometry. Its fixed point set $S_m^2(0)$ is a totally geodesic surface called the horizon. Note that $(\mathbb{R}^3 \setminus B_m^2(0), g_m)$ is an asymptotically flat manifold with one end.

We say that an asymptotically flat end of $(M, g)$ is $C^k$-asymptotic to Schwarzschild of mass $m > 0$ if in some asymptotically flat coordinate system we have that

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2})$$

together with (partial) derivatives of order up to and including $k$.

For ease of exposition we assume that asymptotically flat manifolds have one end, unless explicitly indicated otherwise.

Assume that $R \in L^1(M, g)$. The limit

$$m_{ADM} := \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \left( \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right) \frac{x^i}{|x|}$$

exists. It is called the ADM mass of $(M, g)$. It is independent of the particular choice of asymptotic coordinate system.

The definition of mass is obtained from Hamiltonian formalism of general relativity. It is a conserved quantity under the time evolution of the initial data set under the Einstein flow. (Noether’s principle)

**Exercise 2.1.** Show that the ADM mass of the Schwarzschild manifold $(\mathbb{R}^3 \setminus B_m^2(0), g_m)$ is equal to $m$.

Let $(M, g)$ be an asymptotically flat 3-manifold with one end and with $R \in L^1(M, g)$. Then

$$m_{iso} = \lim_{r \to \infty} \frac{2}{\text{area}_g(S_r)} \left( \text{vol}_g(B_r) - \frac{1}{\sqrt{36\pi}} \text{area}_g(S_r)^{3/2} \right).$$

The expression on the right hand side of (1) was proposed as the “isoperimetric mass” of the initial data set by G. Huisken. It was shown by Fan–Shi–Tam and Miao in [2] that $m_{ADM} = m_{iso}$.

Note that $(36\pi)^{-1/2}\text{area}_g(S_r)^{3/2}$ is the largest amount of Euclidean volume that can be enclosed by an amount $\text{area}_g(S_r)$ of Euclidean area in $\mathbb{R}^3$. It is a fair question to ask how much $g$-volume more can be fit into a coordinate ball.

The following alternative definition of the isoperimetric mass is maybe more geometric.

$$\tilde{m}_{iso} := \limsup_{V \to \infty} \frac{2}{A(V)} \left( V - \frac{1}{\sqrt{36\pi}} A(V)^{3/2} \right).$$

3. Positive mass theorem

**Theorem 3.1** (Schoen-Yau 1979). Let $(M, g)$ be an asymptotically flat manifold with non-negative integrable scalar curvature. The ADM mass of every end of $(M, g)$ is non-negative. If the ADM mass of some end vanishes, then $(M, g)$ is isometric to Euclidean space.

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The proof is by contradiction.

**Step 1:** Reduce to the case where

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2})$$

and where $R > 0$ everywhere.

**Step 2:** For $\Lambda > 0$ sufficiently large, the mean curvature of the boundary of the slab $K \cup \{(x^1, x^2, x^3) \in M \setminus K : |x^3| \leq \Lambda\}$ is non-negative. There exists a non-compact stable minimal surface $\Sigma$ contained in this slab such that $\Sigma \cap \{(x^1, x^2, x^3) : |(x^1, x^2)| \geq R\} = \{(x^1, x^2, u(x^1, x^2)) : |(x^1, x^2)| \geq R\}$ where $u$ is a smooth function on $\{(x^1, x^2) : |(x^1, x^2)| \geq R\}$ with $u = a + O((x^1, x^2)^{-1})$ for some $a \in [-\Lambda, \Lambda]$.

**Step 3:** By Gauss-Bonnet, $\int_{\Sigma} K_{\Sigma} = 2\pi \chi(\Sigma) - 2\pi \leq 0$. On the other hand, from the stability condition

$$\int_{\Sigma} |\nabla f|^2 \geq \int_{\Sigma} (\text{Re}(\nu, \nu) + |h|^2) f^2$$

for all $f \in C^\infty_c(\Sigma)$, the logarithmic cut-off trick (using the quadratic area growth of $\Sigma$), and the Gauss equations

$$R = 2K + |h|^2 - H^2 + 2\text{Re}(\nu, \nu)$$

we find that

$$0 < \frac{1}{2} \int_{\Sigma} R \leq \int_{\Sigma} K_{\Sigma},$$

a contradiction.

### 3.1. Hawking mass

Let $(M, g)$ be a Riemannian 3-manifold. The Hawking mass of a closed 2-surface $\Sigma \subset M$ is defined as

$$m_{\text{Haw}}(\Sigma) = \frac{\sqrt{|\Sigma|}}{(16\pi)^{3/2}} \left(16\pi - \int_{\Sigma} H^2\right)$$

where $H$ is the mean curvature of $\Sigma$.

**Theorem 3.2** (Christodoulou-Yau ’89). Let $(M, g)$ have non-negative scalar curvature and assume that $\Sigma$ is a closed stable constant mean curvature sphere. Then $m_{\text{Haw}}(\Sigma) \geq 0$.

**Theorem 3.3** (Birkhoff’s theorem). The non-flat boundaryless complete rotationally symmetric 3-manifolds with vanishing scalar curvature are given by $(\mathbb{R}^3 \setminus \{0\}, g_m)$ where $m > 0$.

**Proof.** A rotationally symmetric metric has the form $dr^2 + f^2 g_{S^2}$ where $f = f(r)$ is a smooth positive function. We have that

$$R = 2K_{S^2} - (H_{S^2})^2 + |h_{S^2}|^2 + 2\text{Re} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$$

for all $r$ by the Gauss equation. Note that $K_{S^2} = f^{-2}$, $H_{S^2} = 2f'/f$, $|h_{S^2}|^2 = H^2_{S^2}/2 = 2(f')^2/f^2$, and that $H^2_{S^2} = -|h_{S^2}|^2 - \text{Re} \left(\frac{\partial^2}{\partial r^2}, \frac{\partial^2}{\partial r^2}\right)$. Eliminating the Ricci term from these equations and imposing that $R = 0$ we find that

$$0 = 1 - f^2 - 2ff'''.$$

Note that the derivative of $f(1 - f^2)$ is equal to $f'(1 - f^2 - 2ff'')$ and that the Hawking mass of $S^2$ is (essentially) equal to $f(1 - f^2)$. In other words, the Hawking mass is a first integral for the condition that the scalar curvature vanishes. It is not difficult to check that the spheres of symmetry in Schwarzschild have constant Hawking mass.

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3.2. Penrose inequality.

**Theorem 3.4** (Huisken-Ilmanen '98 and Bray '00). Let \((M,g)\) be an asymptotically flat manifold with \(R \in L^1(M,g)\) and such that \(R \geq 0\). Assume that there are no closed minimal surfaces in \((M,g)\) other than the components of the boundary of \(M\). Then

\[
m_{\text{ADM}} \geq \sqrt{\frac{|\partial M|}{16\pi}}.
\]

If equality holds, then \((M,g)\) is isometric to Schwarzschild initial data \((\mathbb{R}^3 \setminus B_\infty(0), g_m)\).

**Theorem 3.5.** Assume that \((M,g)\) is a 3-dimensional asymptotically flat manifold with non-negative scalar curvature. There exists a sequence of isoperimetric regions \(\Omega_i \subset M\) with \(|\Omega_i| \to \infty\).

**References**


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