1. The Group Algebra

Let $G$ be a finite group of order $n$. For each $g \in G$, write $e_g$ (for the moment, we think of these $e_g$'s as simply 'placeholders' with no algebraic significance). Now, define $\mathbb{C}G$ to be the set of all formal sums

$$\sum_{g \in G} z_ge_g, \ z_g \in \mathbb{C}.$$

**Exercise 1.1.** Check that $\mathbb{C}G$ is a complex, $n$-dimensional vector space with basis $\{e_g | g \in G\}$.

This vector space comes equipped with a natural left action of $G$; for $x \in G$ we define

$$x \cdot \sum_{g \in G} z_ge_g := \sum_{g \in G} z_ge_{xg}.$$

**Exercise 1.2.** Check that this is actually a group action.

By linearity, this $G$ action extends to an action of $\mathbb{C}G$ on $\mathbb{C}G$; in other words, we have a multiplication map $\mathbb{C}G \times \mathbb{C}G \to \mathbb{C}G$. We call the vector space above along with this multiplication the **group algebra** $\mathbb{C}G$ of $G$.

**Example 1.3.** Let $G = S_3 = \{1, (12), (13), (23), (123), (132)\}$. Then an arbitrary element of $\mathbb{C}S_3$ looks like

$$z_1e_1 + z_{(12)}e_{(12)} + z_{(13)}e_{(13)} + z_{(23)}e_{(23)} + z_{(123)}e_{(123)} + z_{(132)}e_{(132)}$$

for some choice of complex numbers $z$. The multiplication in $\mathbb{C}S_3$ works as follows:

$$\begin{align*}
(e_1 - 2e_{(12)})(5e_{(12)} + 3e_{(13)} - 7e_{(132)}) &= e_1(5e_{(12)} + 3e_{(13)} - 7e_{(132)}) - 2e_{(12)}(5e_{(12)} + 3e_{(13)} - 7e_{(132)}) \\
&= 5e_1e_{(12)} + 3e_1e_{(13)} - 7e_1e_{(132)} - 10e_{(12)}e_{(12)} - 6e_{(12)}e_{(13)} + 14e_{(12)}e_{(132)} \\
&= 5e_{(12)} + 3e_{(13)} - 7e_{(132)} - 10e_1 - 6e_{(132)} + 14e_{(13)} \\
&= -10e_1 + 5e_{(12)} + 17e_{(13)} - 13e_{(132)}.
\end{align*}$$

2. Group Algebras and Representations

Let $G$ be a finite group again, and let $\rho : G \to GL(V)$ be a representation of $G$ on a finite-dimensional complex vector space $V$. This representation determines an action of $\mathbb{C}G$ on $V$: for $v \in V$ and $\sum_{g \in G} z_ge_g \in \mathbb{C}G$, define

$$\left(\sum_{g \in G} z_ge_g\right) \cdot v := \sum_{g \in G} z_\rho(g)v.$$
Example 2.1. Let $G = S_3$, and let $\rho : S_3 \to GL(V)$ be the standard representation of $S_3$ on the vector space $V$ spanned by the vectors $f_1 - f_2, f_2 - f_3 \in \mathbb{C}^3$ (here $f_1 = (1, 0, 0), f_2 = (0, 1, 0)$ and $f_3 = (0, 0, 1)$). So, $S_3$ acts on $V$ by permuting the indices of these basis vectors. Consider the elements $2(f_1 - f_2) - 3(f_2 - f_3) \in V$ and $e_{(12)} - 2e_{(132)} \in \mathbb{C}S_3$.

Then,

$$
(e_{(12)} - e_{(132)}) \cdot (2(f_1 - f_2) - 3(f_2 - f_3))
= e_{(12)} \cdot (2(f_1 - f_2) - 3(f_2 - f_3)) - 3e_{(132)} \cdot (2(f_1 - f_2) - 3(f_2 - f_3))
= \rho(12)(2(f_1 - f_2) - 3(f_2 - f_3)) - 3\rho(132)(2(f_1 - f_2) - 3(f_2 - f_3))
= 2(f_2 - f_1) - 3(f_1 - f_3) - 6(f_3 - f_1) + 9(f_1 - f_2)
= 10(f_1 - f_2) + 3(f_2 - f_3).
$$

In the other direction, suppose we are given a vector space $V$ on which $\mathbb{C}G$ acts (we say that $V$ is a "$\mathbb{C}G$-module"). This action determines a representation $\rho : G \to GL(V)$ in the following way: let $g \in G$ and let $v \in V$. Then define

$$
\rho(g)v := e_g \cdot v.
$$

Exercise 2.2. Show that the definition above actually gives a representation of $G$ (the key thing to check is that $G$ maps to linear transformations of $V$.)

So, we see that "representations $(\rho, V)$ of $G$" and "$\mathbb{C}G$-modules $V$" are really two descriptions of the same thing. Both terminologies are used interchangeably, and both descriptions have their advantages. For example, in the context of group algebras, certain operations such as tensoring and inducing representations become more intuitive and many of the proofs are cleaner.

Exercise 2.3. Reconcile the ideas of "irreducible representation $(\rho, V)$ of $G$" and "irreducible $\mathbb{C}G$-module $V$". Convince yourself they are the same things.