1) Why can every rotation by $\theta$ in $\mathbb{R}^2$ be represented by a matrix of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

2) Why does rotation by arbitrary angles form a group?

3) In class, we showed that we can realize the change of coordinates $x \mapsto u + v$ and $y \mapsto v$ as the multiplication

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Furthermore, we showed that this gives rise to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which acts on the coefficients of the cubic

$$ax^3 + bx^2y + cxy^2 + dy^3.$$ 

Do a similar construction for generic quartics and generic quintics using the same matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (or just use an invertible matrix of your choosing).
4) The construction described in question 3 shows that

\[
\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

in a certain representation \( \rho : GL_2 \to GL_4 \).

Use Maple (or any software of your choice) to write a program that will determine \( \rho \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) for an arbitrary matrix. Do the same for the representation \( \rho : GL_3 \to GL_6 \), where elements of \( GL_3 \) act on conics in \( \mathbb{P}^2 \).

5) Finish the construction of the representation

\( \rho : S_3 \to GL_2 \)

that we began in class. Recall, in this representation the permutations in \( S_3 \) act on the vertices of an equilateral triangle centered at the origin.