Weak immersions of surfaces with $L^2$-bounded second fundamental form

Lecture 3

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4 Sequences of weak immersions

4.1 Compactness question

Eventually we will be interested in finding a minimizer of the Willmore energy. More specifically, we will look at a minimizing sequence of weak immersions with $L^2$-bounded second fundamental form and ask if it has a limit with respect to some notion of weak convergence and how this limiting object looks like.

Recall that in Section 1.3 we showed that the Willmore energy is invariant under conformal transformations in the target. To get an idea what kind of convergence is the best we can hope for, let us study an example: Assume $\vec{\Phi} \in \mathcal{E}_\Sigma$ immerses a torus in $\mathbb{R}^3$. Figures 1, 2 and 3 show that composing $\vec{\Phi}$ with sequences of conformal diffeomorphisms of $\mathbb{R}^3$ of different types produces sequences $\vec{\Phi}_k$, whose limit does not immerse a torus anymore.

Here, for $a \in \mathbb{R}^m$, the map $i_a \in \text{Conf}(\mathbb{R}^3 \cup \{\infty\})$ denotes the inversion at $a$,

$$i_a : x \mapsto \frac{x - a}{|x - a|^2}.$$

Note that $i_a$ is a diffeomorphism from $B_R(a) \setminus B_\varepsilon(a)$ to $B_{1/R}(a) \setminus B_{1/R}(a)$ for any $0 < \varepsilon < R < \infty$ and thus, for $a \in \mathbb{R}^m \setminus \vec{\Phi}(\Sigma)$,

$$W(i_a \circ \vec{\Phi}) = W(\vec{\Phi}).$$

These examples illustrate that, given a sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$, say with $\sup_k W(\vec{\Phi}_k) < \infty$, we cannot expect that $\vec{\Phi}_k$ has a limit, which still immerses the surface $\Sigma$ (in a weak sense). In fact, we need to compose with conformal transformations $\vec{\Xi}_k \in \text{Conf}(\mathbb{R}^m \cup \{\infty\})$ (with the center of inversion of $\vec{\Xi}_k$

![Figure 1: Dilation. $\vec{\Phi}_k := k \vec{\Phi}$. (Loss of energy and topology.)](image-url)
Figure 2: Inversion. \( \Phi_k := i_{a_k} \circ \Phi \), where \( a_k \to \Phi(\Sigma) \setminus \{0\} \). (Loss of energy.)

Figure 3: Inversion \( \circ \) Dilation. \( \Phi_k := \text{dist}(a_k, \Phi(\Sigma)) \circ i_{a_k} \circ \Phi \), where \( a_k \to \Phi(\Sigma) \setminus \{0\} \). (Loss of energy and topology.)

not being contained in \( \Phi_k(\Sigma) \)) to avoid degeneracies as shown in Figures 1 - 3.

Furthermore, when passing to the limit there might be energy concentration in single points. Figure 4 provides an example of how such a loss of energy in the limit might occur.

To allow composing with conformal transformations along the sequence in order to obtain a reasonable (weak) limiting object is not enough: We also need to make a compactness assumption on the conformal classes induced by \( \Phi_k \). Let \( \Psi_k : (\Sigma, h_k) \to \Sigma \) be the Lipschitz diffeomorphisms from Corollary 3.4 such that

\[ \Psi_k := \Phi_k \circ \Psi_k : (\Sigma, h_k) \to \mathbb{R}^m \]

are conformal, where each \( h_k \) denotes the reference metric of constant curvature and unit volume of the conformal structure induced by \( \Phi_k \). Since \( \Psi_k \)
are elements of the invariance group $\text{Diff}^+(\Sigma)$ of the Willmore functional, we have $W(\tilde{\Psi}_k) = W(\tilde{\Phi}_k)$. We make the following compactness assumption (CA):

The conformal classes $(\Sigma, h_k)$ are contained in a compact subset of $\mathcal{M}_\Sigma$, the moduli space of $\Sigma$.

This assumption is necessary for the following reason: If the conformal classes degenerate, we might not only lose energy but also topology in the limit, which is irreversible and has to be avoided. Such a situation is shown in Figure 5. Note that there is no way to preserve the genus in the limit: One can “save” one hole (e.g. by applying dilations and inversions as in Figure 3), but one will lose the other two ones at the same time.

Before using the previous observations to define a notion of weak convergence, we note the following fact:

If a sequence of $\tilde{\Phi}_k \in \mathcal{E}_\Sigma$ satisfies the compactness assumption (CA), the conformal classes $(\Sigma, h_k)$ of constant curvature and unit volume, induced by $\tilde{\Phi}_k$ resp., satisfy (up to subsequences)

$$h_k \rightarrow h_\infty \text{ in } C^l(\Sigma) \quad \forall l \in \mathbb{N},$$

where $(\Sigma, h_\infty)$ is the limiting conformal structure of constant curvature and unit volume.

**Definition 4.1.** A sequence $\tilde{\Phi}_k \in \mathcal{E}_\Sigma$ satisfying assumption (CA) with $h_k \rightarrow h_\infty$ is called weakly convergent if there exist Lipschitz diffeomorphisms $\Psi_k$ of $\Sigma$, conformal transformations $\tilde{\Xi}_k$ of $\mathbb{R}^m \cup \{\infty\}$ with

$$\tilde{\Phi}_k(\Sigma) \cap \{\text{center of inversion of } \tilde{\Xi}_k\} = \emptyset$$

![Figure 4: Loss of energy, no loss of topology.](image-url)
and finitely many points $a_1, \ldots, a_N \in \Sigma$, called blow-up points such that

$$\vec{\xi}_k := \vec{\Xi}_k \circ \vec{\Phi}_k \circ \vec{\Psi}_k : (\Sigma, h_k) \to \mathbb{R}^m$$

is conformal, and there exists a map $\vec{\xi}_\infty : \Sigma \to \mathbb{R}^m$ such that

i) $\vec{\xi}_\infty$ is conformal from $(\Sigma, h_\infty)$ into $\mathbb{R}^m$;

ii) $\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty$ weakly in $W^{2,2}_{\text{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\})$; (4.1)

iii) $\log |d\vec{\xi}_k|^2 \rightharpoonup \log |d\vec{\xi}_\infty|^2$ weakly in $(L^\infty)^*_\text{loc}(\Sigma \setminus \{a_1, \ldots, a_N\})$; (4.2)

iv) $\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty$ weakly in $W^{1,2} \cap (L^\infty)^*(\Sigma)$. (4.3)

The following lemma shows that the Willmore functional $W$ and the energy functional $I$ are lower semicontinuous under weak convergence.

**Lemma 4.2.** Let $(\Sigma, h_k)$ be a sequence of conformal structures on $\Sigma$, where $h_k$ denotes the associated metric of constant curvature and unit volume. Assume it satisfies assumption (CA), with $h_k \to h_\infty$. 

Figure 5: Loss of energy, topology and conformal class.
Let \( \bar{\xi}_k : (\Sigma, h_k) \to \mathbb{R}^m \) be a sequence of conformal maps in \( E_{\Sigma} \) with

\[
\sup_k \| \xi_k \| < \infty
\]

and \( \bar{\xi}_\infty : (\Sigma, h_\infty) \to \mathbb{R}^m \) a conformal map such that ii) and iii) from Definition 4.1 are satisfied, i.e. there exist \( a_1, \ldots, a_N \in \Sigma \) such that

\[
\bar{\xi}_k \rightharpoonup \bar{\xi}_\infty \text{ weakly in } W^{2,2}_{\text{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\})
\]

and

\[
\log |d\xi_k|^2 \rightarrow \log |d\xi_\infty|^2 \text{ weakly in } (L^\infty)^*_{\text{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\}).
\]

Then for any \( K \subset \Sigma \setminus \{a_1, \ldots, a_N\} \), we have

\[
\int_K |\bar{H}_{\xi_\infty}|^2 d\text{vol}_{\bar{g}_\infty} \leq \liminf_k \int_K |\bar{H}_{\xi_k}|^2 d\text{vol}_{g_k}
\]

and

\[
\int_K |d\bar{n}_{\xi_\infty}|^2 g_{\infty} d\text{vol}_{\bar{g}_\infty} \leq \liminf_k \int_K |d\bar{n}_{\xi_k}|^2 g_k d\text{vol}_{g_k}.
\]

Proof. Since the \( \bar{\xi}_k \)'s are conformal, the mean curvature vector in isothermal coordinates can be written as

\[
\bar{H}_{\bar{\xi}_k} = \frac{1}{2} e^{-2\lambda_k} \Delta \bar{\xi}_k,
\]

where \( \lambda_k := \log |\partial_{x_1} \bar{\xi}_k| \).

Let \( K \subset \Sigma \setminus \{a_1, \ldots, a_N\} \) be compact. Note that, due to (4.5),

\[
\Delta \bar{\xi}_k \rightharpoonup \Delta \bar{\xi}_\infty \quad \text{weakly in } L^2(K).
\]

Moreover, using again (4.5) and applying Rellich/Kondrachov, we obtain for all \( 1 < p < \infty \) a subsequence such that

\[
\partial_{x_1} \bar{\xi}_k \rightharpoonup \partial_{x_1} \bar{\xi}_\infty \quad \text{strongly in } L^p(K).
\]

(4.6) ensures that \( |\partial_{x_1} \bar{\xi}_k| \geq C \) a.e. for \( C > 0 \) independent of \( k \). Thus, we have

\[
e^{-\lambda_k} = \frac{1}{|\partial_{x_1} \bar{\xi}_k|} \rightarrow \frac{1}{|\partial_{x_1} \bar{\xi}_\infty|} = e^{-\lambda_\infty} \quad \text{strongly in } L^p(K).
\]
This, together with (4.9), implies
\[
\tilde{H}_{\xi_k'} \sqrt{vol_{g_{k'}}} = \frac{1}{2} e^{-\lambda_k'} \Delta \tilde{\xi}_{k'} \to \frac{1}{2} e^{-\lambda_{\infty}} \Delta \tilde{\xi}_{\infty} = \tilde{H}_{\xi_{\infty}} \sqrt{vol_{g_{\infty}}}
\]
in \( \mathcal{D}'(K) \).  
(4.10)

But note that \( \Delta \tilde{\xi}_{k'} \) is uniformly bounded in \( L^2(K) \), due to (4.5), and \( e^{-\lambda_{k'}} \) is uniformly bounded in \( L^\infty(K) \), due to (4.6). It follows that \( \tilde{H}_{\xi_k'} \sqrt{vol_{g_{k'}}} \) is uniformly bounded in \( L^2(K) \) and consequently, the convergence in (4.10) is a weak convergence in \( L^2(K) \). Lower semicontinuity of the \( L^2 \)-norm under weak \( L^2 \)-convergence implies the desired result.

To prove (4.8), note that, by \( h_k \to h_{\infty} \), (4.4) implies that
\[
\sup_k \int_{\Sigma} |d\tilde{n}_{\xi_k'}|^2_{h_{\infty}} dvol_{h_{\infty}} < \infty.
\]

Thus, up to subsequences, there exists \( \tilde{n}_{\infty} \in W^{1,2}(\Sigma) \) such that
\[
\tilde{n}_{\xi_k'} \to \tilde{n}_{\infty} \text{ weakly in } W^{1,2}(\Sigma).
\]

We will show that \( \tilde{n}_{\infty} \) equals \( \tilde{n}_{\xi_{\infty}} \) on any compact set \( K \subset \Sigma \setminus \{a_1, \ldots, a_N\} \), which in turn implies (4.8), by lower semicontinuity of weak \( W^{1,2} \)-convergence and \( h_k \to h_{\infty} \) again.

Given any \( p < \infty \), we have, modulo extraction of a subsequence, strong convergence \( \tilde{n}_{\xi_k'} \to \tilde{n}_{\infty} \) in \( L^p(\Sigma) \), by Rellich/Kondrachov for compact manifolds (see [Aub82], Theorem 2.34). After passing to a further subsequence, we can assume that
\[
\tilde{n}_{\xi_k'} \to \tilde{n}_{\infty} \text{ a.e. in } \Sigma.  \tag{4.11}
\]

Note that in isothermal coordinates, we have
\[
\tilde{n}_{\xi_k'} = e^{-2\lambda_{k'}} \ast \left( \partial_{x_1} \tilde{\xi}_{k'} \wedge \partial_{x_2} \tilde{\xi}_{k'} \right). \tag{4.12}
\]

Applying Rellich/ Kondrachov, (4.5) implies that for any \( p < \infty \), there is a further subsequence such that
\[
d\tilde{\xi}_{k'} \to d\tilde{\xi}_{\infty} \text{ strongly in } L^p(K).
\]

By passing to a further subsequence, we obtain
\[
d\tilde{\xi}_{k'} \to d\tilde{\xi}_{\infty} \text{ a.e. in } K.
\]
This implies in particular that
\[ \star \left( \partial_{x_1} \tilde{\xi}_{k'} \wedge \partial_{x_2} \tilde{\xi}_{k'} \right) \to \star \left( \partial_{x_1} \tilde{\xi}_\infty \wedge \partial_{x_2} \tilde{\xi}_\infty \right) \quad \text{a.e. in } K. \quad (4.13) \]

Observe that \( \sup_{k'} \| \lambda_{k'} \|_{W^{1,2}(K)} \leq C \), due to
\[ |\nabla \lambda_{k'}| = \left| \nabla \log \left( \frac{1}{\sqrt{2}} |\nabla \tilde{\xi}_{k'}| \right) \right| \leq C + \frac{|\nabla^2 \tilde{\xi}_{k'}|}{|\nabla \tilde{\xi}_{k'}|}, \]
(4.5) and (4.6), which implies \( \sup_k |\nabla \tilde{\xi}_k| \geq c > 0 \). Consequently, we can assume that
\[ \lambda_{k'} \to \lambda_\infty \quad \text{in } L^p(K) \text{ and a.e. in } K, \quad (4.14) \]
after extraction of subsequences. (4.12), (4.14) and (4.13) imply that
\[ \tilde{n}_{\tilde{\xi}_{k'}} \to e^{-2\lambda_\infty} \star \left( \partial_{x_1} \tilde{\xi}_\infty \wedge \partial_{x_2} \tilde{\xi}_\infty \right) \quad \text{a.e. in } K. \]
(4.11), together with uniqueness of the limit, gives the desired result that \( \tilde{n}_{\tilde{\xi}_\infty} \) and \( \tilde{n}_\infty \) coincide on \( K \).

\[ \square \]

**Theorem 4.3** (Weak almost-closure theorem). Let \( \Phi_k \in \mathcal{E}_\Sigma \) such that
\[ \sup_k \mathbb{I}(\Phi_k) < \infty \quad (4.15) \]
and such that assumption (CA) is satisfied.

Then there exists a weakly converging subsequence of \( \Phi_k \) (in the sense of Definition 4.1).

In the next two subsections we shall prepare the proof of Theorem 4.3, which will be finally given in Subsection 4.4.

### 4.2 Control of the conformal factor

Let \( \tilde{\Phi}_k \in \mathcal{E}_\Sigma \) be a sequence of weak immersions. Corollary 3.4 tells us how to fix a gauge, namely one can find Lipschitz diffeomorphisms \( \Psi_k: (\Sigma, h_k) \to \Sigma \) such that
\[ \tilde{\Psi}_k := \tilde{\Phi}_k \circ \Psi_k: (\Sigma, h_k) \to \mathbb{R}^m \]
are conformal, where each $h_k$ denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\Phi_k$. Denote

$$g_{\Phi_k} = e^{2\alpha_k}h_k.$$ 

A priori the conformal factors $e^{2\alpha_k}$ could go either to $+\infty$ or $0$ as $k \to \infty$. In both cases, the limiting map will not be an element of $E_\Sigma$, in the first case failing the Lipschitz condition and in the second case failing the non-degeneracy of the metric. The question that we want to investigate now is therefore: Can the logarithms of the conformal factors, that are $\alpha_k$, be controlled in the $L^\infty$-norm by $\sup_k \|\bar{\Phi}_k\|$, when we let $k \to \infty$?

The first result is a global bound for $\alpha_k$ in the $L^{2,\infty}(\Sigma)$-norm.

**Theorem 4.4.** Let $\Phi_k \in E_\Sigma$ be a sequence of Lipschitz immersions with $L^2$-bounded second fundamental form such that

$$\sup_k \|\bar{\Phi}_k\| < \infty.$$ 

Let $\Psi_k$ be Lipschitz diffeomorphisms of $\Sigma$ such that

$$\bar{\Psi}_k := \Phi_k \circ \Psi_k : (\Sigma, h_k) \to \mathbb{R}^m$$

are conformal, where each $h_k$ denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\Phi_k$. Furthermore, we make assumption (CA), that is the conformal classes $(\Sigma, h_k)$ are contained in a compact subset of $\mathcal{M}_\Sigma$.

Denote $g_k := \Phi_k^*g_{\mathbb{R}^m}$ and

$$g_k = e^{2\alpha_k}h_k.$$ 

Then

$$\sup_k \|d\alpha_k\|_{L^{2,\infty}(\Sigma)} < \infty.$$ 

**Proof.** Since $g_k = e^{2\alpha_k}h_k$, by (1.52), we have for all $k \in \mathbb{N}$

$$-\Delta_{h_k} \alpha_k = e^{2\alpha_k}K_{g_k} - K_{h_k}.$$ 

(4.16)

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This identity, together with (1.23), gives us the following estimate for $\Delta h_k \alpha_k$ in $L^1_{h_k}(\Sigma)$:

$$
\int_\Sigma |\Delta h_k \alpha_k| \, dv_{h_k} \leq \int_\Sigma e^{2\alpha_k} |K_{g_k}| \, dv_{h_k} + \int_\Sigma |K_{h_k}| \, dv_{h_k}
$$

$$
= \int_\Sigma |K_{g_k}| \, dv_{g_k} + |K_{h_k}| \leq \frac{1}{2} \int_\Sigma |\tilde{\nabla}_{g_k}^2 \phi_k| \, dv_{g_k} + |K_{h_k}|
$$

(4.17)

$$
\leq \frac{1}{2} \sup_k I(\Phi_k) + C =: \tilde{C},
$$

where we used that we chose $h_k$ to be the constant curvature metric of unit volume and $h_k \to h_\infty$. Since $c_k^{-1} g_0 \leq h_k \leq c_k g_0$ and due to $h_k \to h_\infty$, (4.17) gives also the estimate

$$
\|\Delta h_k \alpha_k\|_{L^1_{g_0}(\Sigma)} \leq \tilde{C}
$$

for all $k \in \mathbb{N}$. Applying Theorem 2.6 yields thus the desired uniform bound for $d\alpha_k$ in $L^{2,\infty}_{g_0}(\Sigma)$:

$$
\|d\alpha_k\|_{L^{2,\infty}_{g_0}(\Sigma)} \leq C_{g_0} \|\Delta h_k \alpha_k\|_{L^1_{g_0}(\Sigma)} \leq C_{g_0} \cdot \tilde{C}.
$$

(4.18)

We now shall investigate the evolvement of the logarithms of the conformal factors in the $L^\infty$-norm. This is done locally, wherever the second fundamental form does not concentrate “too much energy”.

**Theorem 4.5.** Let $\Phi_k \in \mathcal{E}_\Sigma$ be a sequence in $\mathcal{E}_\Sigma$ which satisfies

$$
\sup_k I(\Phi_k) < \infty.
$$

Furthermore, we make assumption (CA): the conformal classes $(\Sigma, h_k)$ are contained in a compact subset of the moduli space of $\Sigma$. As before, let $\Psi_k$ be Lipschitz diffeomorphisms of $\Sigma$ such that

$$
\Psi_k := \Phi_k \circ \Psi_k : (\Sigma, h_k) \to \mathbb{R}^m
$$

are conformal, where each $h_k$ denotes the reference metric of constant curvature and unit volume of the conformal structure induced by $\Phi_k$. Moreover, let $\varphi_k$ be a sequence of isothermal charts satisfying

$$
\sup_k \int_{D^2} |\nabla \phi_k \circ \varphi_k|^2 \, dx_1 dx_2 < \frac{8\pi}{3}.
$$

(4.19)
Denote
\[(\vec{\Psi}_k \circ \varphi_k)^* g_{Rm} = e^{2\lambda_k}(dx_1^2 + dx_2^2).\]

Then there exist constants \(c_k \in \mathbb{R}\) such that
\[
\sup_k \|\lambda_k - c_k\|_{L^\infty(\Omega)} \leq C_{\Omega},
\]
(4.20)
for any set \(\Omega \subset D^2\).

The following example shows that in general we cannot control the \(\lambda_k\) in \(L^\infty\) when deleting the constants \(c_k\) in (4.20): Compose the weak immersions \(\vec{\Psi}_k\) with dilations \(s_k \cdot \) in \(\mathbb{R}^m\), for some \(s_k \in \mathbb{R}\). Due to the conformal invariance of the Willmore functional, we have \(\mathcal{I}(\vec{\Psi}_k) = \mathcal{I}(s_k \vec{\Psi}_k)\). In contrast, the dilations are reflected in the logarithms of the conformal factors, that is \(\lambda_{s_k \vec{\Psi}_k} = \lambda_{\vec{\Psi}_k} + \log s_k\).

Note that this example also shows that the constants \(c_k\) are a priori not controlled, when \(k \to \infty\).

**Proof of Theorem 4.5.** Assuming (4.19), for each \(k \in \mathbb{N}\), we can apply Hélein’s lifting theorem 3.2 to obtain the existence of \(\vec{f}_1^k\) and \(\vec{f}_2^k\) in \(W^{1,2}(D^2, S^{m-1})\) such that
\[\vec{n}_k := \vec{n}_{\vec{\Psi}_k \circ \varphi_k} = \star (\vec{f}_1^k \land \vec{f}_2^k),\]
and
\[
\int_{D^2} \sum_{i=1}^2 |\nabla \vec{f}_i^k|^2 \, dx_1 \, dx_2 \leq C \int_{D^2} |\nabla \vec{n}_k|^2 \, dx_1 \, dx_2,
\]
where \(\vec{n}_k := \vec{n}_{\vec{\Psi}_k \circ \varphi_k}\).

Using the notation introduced in Section 1.1, the factors \(\lambda_k\) satisfy the following equation:
\[
\Delta \lambda_k = -\langle \nabla \perp \vec{f}_1^k, \nabla \vec{f}_2^k \rangle.
\]
(4.21)
This can seen by observing that, by (1.45), the latter equation holds for the frame \((\vec{f}_1^k, \vec{f}_2^k)\) being replaced by the frame \((\vec{e}_1^k, \vec{e}_2^k)\) given by
\[\vec{e}_k^i := e^{-\lambda_k} \partial_{x_i} (\vec{\Psi}_k \circ \varphi_k).\]
Furthermore, the change of gauge formula (1.60) leaves the right hand side of (4.21) invariant. Observe that the right hand side of (4.21) is a sum of Jacobians, more specifically
\[-\langle \nabla \perp \vec{f}_1^k, \nabla \vec{f}_2^k \rangle = \langle \partial_{x_2} \vec{f}_1^k, \partial_{x_1} \vec{f}_2^k \rangle - \langle \partial_{x_1} \vec{f}_1^k, \partial_{x_2} \vec{f}_2^k \rangle\]
\[
= \sum_{j=1}^{m} \partial_{x_2} f_{1,j}^k \cdot \partial_{x_1} f_{2,j}^k - \partial_{x_1} f_{1,j}^k \cdot \partial_{x_2} f_{2,j}^k.
\]

Let \( \mu_k \) be the solution to
\[
\begin{aligned}
\Delta \mu_k &= \sum_{j=1}^{m} \partial_{x_2} f_{1,j}^k \cdot \partial_{x_1} f_{2,j}^k - \partial_{x_1} f_{1,j}^k \cdot \partial_{x_2} f_{2,j}^k \quad \text{on } D^2 \\
\mu_k &= 0 \quad \text{on } \partial D^2.
\end{aligned}
\]

Wente’s Theorem 2.8 gives us the estimate
\[
\| \nabla \mu_k \|_{L^2(D^2)} + \| \mu_k \|_{L^\infty(D^2)} \leq C \sum_{j=1}^{m} \int_{D^2} \left( |\nabla f_{1,j}^k|^2 + |\nabla f_{2,j}^k|^2 \right) dx_1 dx_2
\]
\[
\leq C \int_{D^2} |\nabla \vec{n}_k|^2 dx_1 dx_2 \leq C,
\]

where we used (4.19) in the last step such that the constant \( C \) is independent of \( k \).

We now consider the harmonic rest \( \nu_k := \lambda_k - \mu_k \). Due to Theorem 4.4, which gives a global estimate of \( d\alpha_k \) in \( L^{2,\infty}(\Sigma) \), and the strong convergence \( h_k \to h_\infty \), we have a uniform bound
\[
\sup_k \| \nabla \lambda_k \|_{L^{2,\infty}(D^2)} \leq C.
\]

Together with (4.22) (and the fact that \( L^2 \hookrightarrow L^{2,\infty} \) continuously), this yields \( \nu_k \in L^{2,\infty}(D^2) \) and
\[
\sup_k \| \nabla \nu_k \|_{L^{2,\infty}(D^2)} \leq C.
\]

From (2.5), we know that
\[
\| \nabla \nu_k \|_{L^p(D^2)} \leq C_p \| \nabla \nu_k \|_{L^{2,\infty}(D^2)}
\]
for all \( p < 2 \). Applying Poincaré’s inequality yields
\[
\| \nu_k - \bar{\nu}_k \|_{L^p(D^2)} \leq C_p \| \nabla \nu_k \|_{L^p(D^2)},
\]

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where $\bar{\nu}_k$ denotes the average of $\nu_k$ on $D^2$.

Since $W^{1,p}(D^2) \hookrightarrow W^{1-\frac{1}{p'},p}(\partial D^2) \hookrightarrow L^1(\partial D^2)$, we obtain
\[
\sup_k \|\nu_k - \bar{\nu}_k\|_{L^1(\partial D^2)} \leq C.
\]

Using the Poisson representation formula for harmonic functions on $D^2$ yields
\[
\|\nu_k - \bar{\nu}_k\|_{C^1(\Omega)} \leq C_\Omega \|\nu_k - \bar{\nu}_k\|_{L^1(\partial D^2)}
\]
for any $\Omega \subset D^2$ and thus
\[
\sup_k \|\nu_k - \bar{\nu}_k\|_{L^\infty(\Omega)} \leq \tilde{C}_\Omega.
\]

Combining this result with (4.22) gives us
\[
\sup_k \|\lambda_k - c_k\|_{L^\infty(\Omega)} \leq \hat{C}_\Omega,
\]
for any set $\Omega \subset D^2$, where $c_k := \bar{\nu}_k$.

\[\square\]

4.3 The monotonicity formula and consequences

As always, let $\Sigma$ be a smooth closed oriented surface.

**Lemma 4.6 (Monotonicity formula).** Let $\bar{\Phi} \in \mathcal{E}_\Sigma$ be a Lipschitz immersion with $L^2$-bounded second fundamental form. Denote by $M := \bar{\Phi}(\Sigma)$ the immersed surface.

Then for any point $x_0 \in \mathbb{R}^m$ and any $0 < t < T < \infty$, we have
\[
\frac{\text{Area}(M \cap B_T(x_0))}{T^2} - \frac{\text{Area}(M \cap B_t(x_0))}{t^2} \geq -\frac{1}{4} \int_{M \cap (B_T(x_0) \setminus B_t(x_0))} |\bar{H}|^2 \, d\text{vol}_g
\]
\[
- \frac{1}{T^2} \int_{M \cap B_T(x_0)} \langle x - x_0, \bar{H} \rangle \, d\text{vol}_g - \frac{1}{t^2} \int_{M \cap B_t(x_0)} \langle x - x_0, \bar{H} \rangle \, d\text{vol}_g.
\]

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Definition 4.7. Let $\tilde{\Phi} \in \mathcal{E}_\Sigma$. For $x_0 \in \mathbb{R}^m$, define the density $\theta_{x_0} \in \mathbb{N}$ at $x_0$ as
\[
\theta_{x_0} := \lim_{t \to 0^+} \frac{1}{\pi t^2} \text{Area} \left( \tilde{\Phi}(\Sigma) \cap B_t(x_0) \right),
\]
where $\text{Area}(\tilde{\Phi}(\Sigma) \cap B_t(x_0))$ is the area of the intersection of $\tilde{\Phi}(\Sigma)$ and the ball $B_t(x_0)$. Whenever the limit exists.

To simplify notation in the proofs of the following corollaries, we introduce for fixed $x_0 \in \mathbb{R}^m$ the quantities
\[
W(t) := \int_{M \cap B_t(x_0)} |\tilde{H}|^2 \, dvol_g,
\]
\[Y(t) := \frac{1}{t^2} \text{Area} \left( \tilde{\Phi}(\Sigma) \cap B_t(x_0) \right).
\]

Corollary 4.8 (Existence of the density). Let $\tilde{\Phi} \in \mathcal{E}_\Sigma$. Then the density $\theta_{x_0}$ exists for every point $x \in \mathbb{R}^m$.

Proof. Let $x_0 \in \mathbb{R}^m$ be an arbitrary point.

Using the Cauchy-Schwarz inequality, we obtain for any $t > 0$:
\[
\frac{1}{t^2} \int_{M \cap B_t(x_0)} \langle x - x_0, \tilde{H} \rangle \, dvol_g
\leq \frac{1}{t} \left( \int_{M \cap B_t(x_0)} |x - x_0|^2 \, dvol_g \right)^{1/2} \left( \int_{M \cap B_t(x_0)} |\tilde{H}|^2 \, dvol_g \right)^{1/2}
\]
\[
\leq Y(t)^{1/2} W(t)^{1/2} \leq \frac{1}{2} \left( Y(t) + W(t) \right),
\]
where we again denote $M := \tilde{\Phi}(\Sigma)$. Since $W(\tilde{\Phi}) < \infty$, we have $W(t) \to 0$ for $t \to 0^+$ and thus, as a direct consequence of Lemma 4.6, we obtain for $0 < t < T < \infty$,
\[
(1 - o_t(1)) Y(t) \leq Y(T) \left( 1 + o(T) \right) + o_{t,T}(1).
\]
Hence,
\[
\lim_{t \to 0^+} Y(t) = \theta_{x_0}
\]
exists. \qed
Corollary 4.9 (Li-Yau inequality). Let \( \vec{\Phi} \in \mathcal{E}_\Sigma \). Then for any \( x_0 \in \mathbb{R}^m \),

\[
\theta_{x_0} \leq \frac{1}{4\pi} \int_{\Sigma} |\vec{H}|^2 \, d\text{vol}_g.
\] (4.27)

**Proof.** Let \( x_0 \in \mathbb{R}^m \) be an arbitrary point. Note that for \( T > 0 \),

\[
\frac{1}{T^2} \int_{M \cap B_T(x_0)} \langle x - x_0, \vec{H} \rangle \, d\text{vol}_g \leq \frac{1}{T^2} \left( \| \vec{\Phi} \|_{L^\infty(\Sigma)} + |x_0| \right) \text{Area}(\vec{\Phi}(\Sigma))^{1/2} W(\vec{\Phi})^{1/2}
\] (4.28)

\[
\rightarrow 0,
\]

with \( M := \vec{\Phi}(\Sigma) \). Moreover,

\[
\frac{1}{T^2} \text{Area}(\vec{\Phi}(\Sigma) \cap B_T(x_0)) \leq \frac{1}{T^2} \text{Area}(\vec{\Phi}(\Sigma)) \rightarrow 0.
\] (4.29)

Using Lemma 4.6, (4.26), (4.28) and (4.29) gives

\[
\frac{1}{4} \int_{\Sigma} |\vec{H}|^2 \, d\text{vol}_g \geq (1 - o_t(1))Y(t) + o_{1/T}(1).
\]

Hence, if we let \( T \rightarrow \infty \), we obtain

\[
\theta_{x_0} = \lim_{t \rightarrow 0^+} \frac{1}{\pi} Y(t) \leq \frac{1}{4\pi} \int_{\Sigma} |\vec{H}|^2 \, d\text{vol}_g.
\]

\[\square\]

Corollary 4.10. Let \( \vec{\Phi} \in \mathcal{E}_\Sigma \) and

\[
W(\vec{\Phi}) < 8\pi.
\] (4.30)

Then the immersion \( \vec{\Phi} \) is in fact an embedding.

**Proof.** For every point \( x_0 \in \mathbb{R}^m \), the density \( \theta_{x_0} \) exists and is an element of \( \mathbb{N} \). If (4.30) holds true, Corollary 4.9 implies that \( \theta_{x_0} \in \{0, 1\} \). This means that \( \vec{\Phi} \) is an embedding. \[\square\]
**Corollary 4.11.** Let $\Phi \in \mathcal{E}_\Sigma$. For any $x_0 \in \Sigma$ and $t > 0$,

$$\text{Area}(\Phi(\Sigma) \cap B_t(x_0)) \leq \frac{3}{2} W(\Phi)t^2. \quad (4.31)$$

**Proof.** By Lemma 4.6, (4.26), (4.28) and (4.29) we have

$$\frac{1}{2} Y(t) \leq \left(\frac{1}{4} + \frac{1}{2}\right) W(\Phi) + o_{1/T}(1).$$

For $T \to \infty$, the result follows. \hfill \Box

**Corollary 4.12.** Let $\Phi \in \mathcal{E}_\Sigma$ and $x_0 \in \mathbb{R}^m$. If $\theta_{x_0} \neq 0$, then for any $T > 0$ we have

$$\text{Area}(\Phi(\Sigma) \cap B_T(x_0)) \geq \frac{2\pi}{3} T^2 - \frac{T^2}{2} \int_{M \cap B_T(x_0)} |\vec{H}|^2 \text{dvol}_g. \quad (4.32)$$

**Proof.** By Lemma 4.6, we obtain for $T > 0$,

$$\pi \theta_{x_0} = \lim_{t \to 0^+} Y(t) \leq Y(T) + \frac{1}{4} W(T) + \frac{1}{T^2} \int_{M \cap B_T(x_0)} \langle x - x_0, \vec{H} \rangle \text{dvol}_g$$

$$\leq \frac{3}{2} Y(T) + \frac{3}{4} W(T),$$

where we used (4.26) in the last step. Hence, if $\theta_{x_0} \geq 1$, then

$$\text{Area}(\Phi(\Sigma) \cap B_T(x_0)) \geq \frac{2\pi}{3} T^2 - \frac{T^2}{2} W(T).$$

\hfill \Box

### 4.4 Proof of the almost-weak closure theorem

**Proof of Theorem 4.3.** Let $\Phi_k \in \mathcal{E}_\Sigma$ be a sequence satisfying the assumptions of Theorem 4.3. Let $\Psi_k$ be diffeomorphisms of $\Sigma$ such that $\Psi_k := \Phi_k \circ \Psi_k: (\Sigma_k, h_k) \to \mathbb{R}^m$ is conformal for any $k \in \mathbb{N}$, where $h_k$ is the metric of...
constant curvature and unit volume of the conformal structure induced by \( \Phi_k \).

Define for each \( k \in \mathbb{N} \) and \( x \in \Sigma \) a number \( \rho_{k,x} > 0 \) by
\[
\rho_{k,x} := \inf \left\{ \rho > 0 \text{ s.t. } \int_{B_{\rho h_k}^k(x)} |d\tilde{n}_{\Phi_k} h_k| h_k^2 \, dvol_{h_k} \geq \frac{8\pi}{3} \right\}.
\]
and
\[
\rho_{\infty,x} := \liminf_{k \to \infty} \rho_{k,x}.
\]
(4.34)

Step 1: The case when there are no concentration points. We first want to investigate the case in which the second fundamental form does not concentrate anywhere, i.e. we assume that
\[
\rho_{\infty} := \inf_{x \in \Sigma} \rho_{\infty,x} > 0.
\]
(4.35)

We will prove that under this assumption, no blow-up points occur such that the limit \( \tilde{x} \) is in \( E_\Sigma \).

Step 1a): Translating and dilating \( \tilde{\psi}_k \) to control the conformal factors in \( L^\infty \) and to bound the image. Assumption (4.35) gives us
\[
\int_{B_{\rho_{\infty,x}/2}(x)} |d\tilde{n}_{\tilde{\Phi}_k} h_k| h_k^2 \, dvol_{h_k} < \frac{8\pi}{3}
\]
(4.36)
for all \( x \in \Sigma \) and large enough \( k \geq k_x \). By conformal invariance of the Dirichlet energy, (4.36) implies that assumption (4.19) is satisfied for any sequence of conformal charts \( \{\varphi_k^x\}_{k \geq k_x} \) around an arbitrary point \( x \in \Sigma \) with \( \varphi_k^x(D^2) = B_{\rho_{\infty,x}/2}(x) \). Since we further suppose (4.15) and condition (CA), we are in the assumptions of Theorem 4.5. Thus for any \( \Omega \in D^2 \), we have
\[
\sup_{x,k \geq k_x} \|\lambda_k^x - c_k^x\|_{L^\infty(\Omega)} \leq C_\Omega,
\]
(4.37)
where \( \lambda_k^x \) (\( c_k^x \) resp.) are the logarithms of the conformal factors (the obtained constants resp.) in the charts \( \varphi_k^x \).
By compactness of $\Sigma$, we can extract a finite subcovering \( \{ B_{h^\infty/\rho}(x_i) \}_{i=1,\ldots,n} \) from \( \{ B_{h_\rho}(x) \}_{x \in \Sigma} \). Then, since \( h_k \to h_\infty \), we can assume that the \( n \) balls \( \{ B_{h^k}(x_i) \}_{i=1,\ldots,n} \) also cover \( \Sigma \), for \( k \in \mathbb{N} \) large enough.

We denote \( \lambda_k = \alpha_k + \sigma_k \) for \( h_k = e^{2 \sigma_k}(dx_1^2 + dx_2^2) \) and \( \vec{\Psi}_k \|_g_{\mathbb{R}^m} = e^{2 \alpha_k}h_k \).

Again due to the strong convergence \( h_k \to h_\infty \), there is a uniform bound

\[
\max_{i=1,\ldots,n} \| \sigma_{x_i}^k \|_{L^\infty(B_{h^k_\rho}(x_i))} \leq M.
\]

This and (4.37) imply for all \( k \geq k_0 := \max\{k_{x_1},\ldots,k_{x_n}\} \) and \( i = 1,\ldots,n \),

\[
\| \alpha_k - c_{x_i}^k \|_{L^\infty(B_{h^k_\rho}(x_i))} \
\leq \| \lambda_{x_i}^k - c_{x_i}^k \|_{L^\infty(B_{h^k_\rho}(x_i))} + \| \sigma_{x_i}^k \|_{L^\infty(B_{h^k_\rho}(x_i))}
\leq M + C\Omega =: \tilde{C},
\]

where \( \Omega \subset D^2 \) is chosen in such a way that \( \varphi_{x_i}^k(\Omega) \supset B_{h_\infty/\rho}(x_i) \) for all \( i = 1,\ldots,n \).

As a result of (4.38), observe that if \( B_{h^k_\rho}(x_i) \cap B_{h^k_\rho}(x_j) \neq \emptyset \), we have

\[
|c_{x_i}^k - c_{x_j}^k| \leq 2\tilde{C}.
\]

Since \( \Sigma \) is path-connected, this yields

\[
\sup_{k \geq k_{0,i,j}} |c_{x_i}^k - c_{x_j}^k| \leq n\tilde{C}.
\]

Next, we compose each \( \vec{\Psi}_k \) with a translation and dilation, conformal transformations in \( \mathbb{R}^m \), in the following way: Define

\[
\vec{\Psi}_k := e^{-c_k} \left( \vec{\Psi}_k - \vec{\Psi}_k(x_0) \right)
\]

for some \( x_0 \in \Sigma \) and \( c_k := c_{x_1}^k \).

The Willmore energy does not change, i.e. we have

\[
W(\vec{\Psi}_k) = W(\vec{\Phi}_k) = W(\vec{\Phi}_k).
\]
What we have achieved by dilating, however, is that $\tilde{\alpha}_k$, the logarithms of the conformal factors of the new conformal immersions $\tilde{\Psi}_k$, are uniformly bounded in $L^\infty(D^2)$:

We have $\tilde{\alpha}_k = \alpha_k - c_k$, and thus for $i = 1, \ldots, n$ and all $k \geq k_0$,

$$\|\tilde{\alpha}_k\|_{L^\infty(B^h_{\rho_{\infty}/4}(x_i))} \leq \|\alpha_k - c_k\|_{L^\infty(B^h_{\rho_{\infty}/4}(x_i))} + \|c_k - c_k\|_{L^\infty(B^h_{\rho_{\infty}/4}(x_i))}$$

$$\leq (1 + n) \tilde{C},$$

by (4.38) and (4.39). Since $\{B^h_{\rho_{\infty}/4}(x_i)\}_{i=1,\ldots,n}$ cover $\Sigma$, this yields

$$\sup_{k \geq k_0} \|\tilde{\alpha}_k\|_{L^\infty(\Sigma)} \leq (1 + n) \tilde{C} =: \hat{C}. \quad (4.41)$$

Further, we performed the translation in order to bound the image of $\tilde{\Psi}_k$ uniformly: $\tilde{\Psi}_k$ maps $x_0 \in \Sigma$ to $0 \in \mathbb{R}^m$ and for any point $y \in \Sigma$, we have

$$|\tilde{\Psi}_k(y) - \tilde{\Psi}_k(x_0)| \leq \int_{x_0}^{y} e^{\tilde{\alpha}_k} dl_{h_k} \leq C,$$

(4.42)

where the constant is independent of $k \in \mathbb{N}$, since $\sup_k \|\tilde{\alpha}_k\|_{L^\infty(\Sigma)}$ and $\sup_k \text{diam}(\Sigma, h_k)$ are finite. It follows that

$$\tilde{\Psi}_k(\Sigma) \subset B_C(0) \quad \text{for all } k \in \mathbb{N}.$$

**Step 1b): Weak $W^{2,2}$-convergence of $\tilde{\Psi}_k$.** Let $\varphi_k$ be a sequence of conformal charts and denote the logarithms of the conformal factors of $\tilde{\Psi}_k \circ \varphi_k$ as $\tilde{\lambda}_k$. Then, by (1.42) the mean curvature vector can be written as

$$\tilde{H}_k = \frac{e^{-2\tilde{\lambda}_k}}{2} \Delta \left( \tilde{\Psi}_k \circ \varphi_k \right)$$

and for all $k \in \mathbb{N},$

$$\left\|\Delta \left( \tilde{\Psi}_k \circ \varphi_k \right) \right\|^2_{L^2(D^2)} = \frac{1}{4} \int_{D^2} e^{4\tilde{\lambda}_k} |\tilde{H}_k|^2 dx_1 dx_2$$

$$\leq \frac{1}{4} \|e^{\tilde{\lambda}_k}\|_{L^\infty(D^2)} \int_{\varphi_k(D^2)} |\tilde{H}_k|^2 dvol_{g_{\tilde{\Psi}_k}} \leq \frac{1}{4} \|e^{\tilde{\lambda}_k}\|_{L^\infty(D^2)} W(\tilde{\Phi}_k) \leq C. \quad (4.43)$$

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Moreover, for all \( k \in \mathbb{N} \) we have
\[
\left\| \nabla \left( \tilde{\Psi}_k \circ \varphi_k \right) \right\|_{L^2(D^2)}^2 = 2 \int_{D^2} e^{2\lambda_k} dx_1 dx_2 \leq C. \tag{4.44}
\]
(4.43) and (4.44) imply that
\[
\sup_k \| \tilde{\Psi}_k \circ \varphi_k \|_{W^{2,2}(D^2)} < \infty. \tag{4.45}
\]
To see this, let \( \mu_k \) be the solution to
\[
\begin{cases}
\Delta \mu_k = \Delta \left( \tilde{\Psi}_k \circ \varphi_k \right) & \text{on } D^2 \\
\mu_k = 0 & \text{on } \partial D^2.
\end{cases}
\]
Then
\[
\| \mu_k \|_{W^{2,2}(D^2)} \leq C \left\| \Delta \left( \tilde{\Psi}_k \circ \varphi_k \right) \right\|_{L^2(D^2)} \leq C \tag{4.46}
\]
for all \( k \in \mathbb{N} \). For the harmonic rest \( \nu_k := \tilde{\Psi}_k \circ \varphi_k - \mu_k \), we get for \( l = 1, 2 \), all \( k \in \mathbb{N} \) and \( \Omega \subset D^2 \),
\[
\| \nabla^l \nu_k \|_{L^2(\Omega)} \leq C_{l,\Omega} \cdot \| \nu_k \|_{L^1(\partial D^2)} \leq \bar{C}_{l,\Omega}, \tag{4.47}
\]
using \( W^{1,2}(D^2) \hookrightarrow L^1(\partial D^2) \), the estimates (4.44) and (4.46) and the Poisson representation formula for harmonic functions on \( D^2 \). (4.46) and (4.47) imply the desired result (4.45).

Due to the strong convergence \( h_k \to h_\infty \), (4.45) implies
\[
\sup_k \| \tilde{\Psi}_k \|_{W^{2,2}(\Sigma)} < \infty. \tag{4.48}
\]
Thus, we can extract a subsequence \( \tilde{\Psi}_{k'} \) such that
\[
\tilde{\Psi}_{k'} \rightharpoonup \tilde{\xi}_\infty \quad \text{weakly in } W^{2,2}(\Sigma) \tag{4.49}
\]
for some \( \tilde{\xi}_\infty \in W^{2,2}(\Sigma) \).
Step 1c): $\tilde{\xi}_\infty$ is conformal and $\log |d\tilde{\Psi}_k|^2 \xrightarrow{\ast} \log |d\tilde{\xi}_\infty|^2$ in $(L^\infty)^*(\Sigma)$. Applying Rellich/ Kondrachov, (4.49) implies that for any $p < \infty$, there is a further subsequence, also denoted by $\tilde{\Psi}_k'$, such that

$$d\tilde{\Psi}_k' \to d\tilde{\xi}_\infty$$ strongly in $L^p(\Sigma)$.

By passing to a further subsequence, again denoted by $\tilde{\Psi}_k'$, we obtain

$$d\tilde{\Psi}_k' \to d\tilde{\xi}_\infty \quad \text{a.e. in } \Sigma. \quad (4.50)$$

This implies that in any sequence of conformal charts, we have for $i, j = 1, 2$,

$$e^{2\tilde{\lambda}_k'} \delta_{ij} = \partial_{x_i} \tilde{\Psi}_k' \partial_{x_j} \tilde{\Psi}_k' \to \partial_{x_i} \tilde{\xi}_\infty \partial_{x_j} \tilde{\xi}_\infty \quad \text{a.e. in } D^2.$$

This yields

$$\partial_{x_i} \tilde{\xi}_\infty \partial_{x_j} \tilde{\xi}_\infty = e^{2\tilde{\lambda}_\infty} \delta_{ij}, \quad (4.51)$$

for $\tilde{\lambda}_\infty := \frac{1}{2} \log \frac{1}{2} |d\tilde{\xi}_\infty|^2$. As a result $\tilde{\xi}_\infty$ is conformal.

Denoting $g_{\tilde{\Psi}_k} = e^{2\tilde{\alpha}_k} h_k$ and $g_{\tilde{\xi}_\infty} = e^{2\tilde{\alpha}_\infty} h_\infty$, we can use (4.41), (4.50) and the dominated convergence theorem to conclude that

$$\tilde{\alpha}_k' \xrightarrow{\ast} \tilde{\alpha}_\infty \quad \text{weakly* in } (L^\infty(\Sigma))^*.$$

Step 2: The general case. We consider the general case and drop assumption (4.35), which means that we allow concentration of energy of the second fundamental form. This will imply the occurrence of blow-up points.

Step 2a): Detecting the concentration points. For given $k \in \mathbb{N}$, the collection $\{B_{\rho_{k,x}}(x)\}_{x \in \Sigma}$ forms a Besicovitch covering of $\Sigma$, where $\rho_{k,x}$ was defined in (4.33). The Besicovitch covering theorem ([Mat95]) gives a sub-covering $\{B_{\rho_{k,x}}(x_k^i)\}_{i \in I_k}$ such that any point in $\Sigma$ is covered by at most $c_\Sigma \in \mathbb{N}$ balls, where $c_\Sigma$ does not depend on $k \in \mathbb{N}$. In fact, $I_k$ is finite and its cardinality uniformly bounded in $k$ since

$$|i \in I_k| \cdot \frac{8\pi}{3} \leq \sum_{i \in I_k} \int_{B_{\rho_{k,x}}(x_k^i)} |d\tilde{\Psi}_k|_{h_k}^2 dvol_{h_k}$$

$$= \int_{\Sigma} |i \in I_k : x \in B_{\rho_{k,x}}(x_k^i)| |d\tilde{\Psi}_k|_{h_k}^2 dvol_{h_k}$$

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\[ c \int_\Sigma |dn \hat{\Psi}_k|^2 dv_{h_k} = c \Im(\Phi_k) \leq C. \]

Thus, we can extract a subsequence such that \( I \) is independent of \( k \) (and finite) and such that for all \( i \in I \),

\[ x_k^i \to x^i_\infty, \quad (4.52) \]

\[ \rho_{k,i} \to \rho_{\infty,i} \quad (4.53) \]

as \( k \to \infty \), for some \( x^i_\infty \in \Sigma \) and \( \rho_{\infty,i} \geq 0 \). Let

\[ J := \{ i \in I \text{ s.t. } \rho_{\infty,i} = 0 \} \quad \text{and} \quad I_0 := I \setminus J. \]

It is clear that \( \bigcup_{i \in I_0} B_{h_\infty}^{h_{\rho_{\infty,i}}} (x^i_\infty) \) covers \( \Sigma \). Note that the balls \( B_{h_\infty}^{h_{\rho_{\infty,i}}} (x^i_\infty) \) are strictly convex: this holds if \( K_{h_\infty} \leq 0 \) and if \( K_{h_\infty} = 1 \) we assume w.l.o.g. that \( \rho_{\infty,i} < \frac{\pi}{2} \). Consequently, the points in \( \Sigma \) which are not contained in the union of the finitely many open balls cannot accumulate and therefore are isolated and hence finite:

\[ \{ a_1, \ldots, a_N \} := \Sigma \setminus \left( \bigcup_{i \in I_0} B_{h_\infty}^{h_{\rho_{\infty,i}}} (x^i_\infty) \right). \quad (4.54) \]

**Step 2b): Applying Step 1a away from the concentration points.**

For arbitrary \( i_0 \in I_0 \), choose \( s_{i_0} < \rho_{\infty,i_0} \). Note that for \( k \) large enough, \( B_{s_{i_0}}^{h_{s_{i_0}}} (x^i_\infty) \subset B_{h_{\rho_{\infty,i_0}}}^{h_k} (x^i_k) \) because \( \rho_{k,i_0} \to \rho_{\infty,i_0} \) and \( x^i_k \to x^i_\infty \). Therefore we can assume that for all \( k \in \mathbb{N} \),

\[ \int_{B_{s_{i_0}}^{h_{s_{i_0}}} (x^i_\infty)} |dn \hat{\Psi}_k|^2 dv_{h_k} < \frac{8\pi}{3}. \]

Theorem 4.5 gives a constant \( c_k \) such that

\[ \sup_k \| \alpha_k - c_k \|_{L^\infty(B_r(x^i_\infty))} \leq C_r \quad (4.55) \]

for any \( r < s_{i_0} \). Define

\[ \bar{\Psi}_k := \frac{1}{e^{-c_k}} \left( \psi_k - \Psi_k(x_0) \right) \]

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for some \( x_0 \in \Sigma \). Let \( K \subseteq \Sigma \setminus \{a_1, \ldots, a_N\} \) be any compact set. Note that, due to (4.54),

\[
\rho_{\infty} := \inf_{x \in K} \rho_{\infty, x} > 0,
\]

where \( \rho_x \) was defined in (4.34). Thus, we can apply Step 1a to any such compact \( K \) (and since \( K \subset \Sigma \setminus \bigcup_{i=1}^N B_\delta(a_i) \) for some \( \delta > 0 \), we restrict to compact sets of the latter form). For \( \delta < \inf_{i \in I_0} \rho_{\infty, i} \), we restrict to compact sets of the latter form).

For \( \delta < \inf_{i \in I_0} \rho_{\infty, i} \), we obtain

\[
\sup_k \|\tilde{\alpha}_k\|_{L^\infty(\Sigma \setminus \bigcup_{i=1}^N B_\delta(a_i))} < C_{\delta},
\]

for a constant \( C_{\delta} \) depending on \( \delta \). Note that (4.57) implies, similarly as in (4.42), that

\[
\tilde{\Psi}_k(\Sigma \setminus \bigcup_{i=1}^N B_\delta(a_i)) \subset B_{C_{\delta}}(0)
\]

for \( k \in \mathbb{N} \).

**Step 2c): Using inversions to control \( \tilde{\Psi}_k(\Sigma) \).** The previous results do not give estimates on what happens in the balls \( B_\delta(a_i) \). In order to get an area control on the limiting map, we would like to improve (4.58) and have the entire image \( \tilde{\Psi}_k(\Sigma) \) contained in a ball.

Since \( \tilde{\Psi}_k(B_\delta(a_i)) \) could degenerate to infinity as \( k \to \infty \), the strategy is to bring this back to a ball around 0 by using inversions. If we can find a ball \( B_r(p_0) \) such that for all \( k \in \mathbb{N} \), \( \tilde{\Psi}_k(\Sigma) \cap B_r(p_0) = \emptyset \), then composing the maps \( \tilde{\Psi}_k \) with the inversion \( i: x \mapsto \frac{x-p_0}{|x-p_0|^2} \) yields \( i \circ \tilde{\Psi}_k(\Sigma) \subseteq B_{1/r}(0) \), as desired.

The existence of such \( p_0 \in B_1(0) \subset \mathbb{R}^m, r > 0 \) is given by the following lemma, which follows from the monotonicity formula.

**Lemma 4.13.** Let \( \Phi_k \in \mathcal{E}_\Sigma \) with

\[
\sup_k W(\Phi_k) < \infty.
\]

Then there exists \( p_0 \in \mathbb{R}^m \) and \( r < 1 - |p_0| \) such that

\[
\Phi_k(\Sigma) \cap B_r(p_0) = \emptyset \quad \text{for all} \quad k \in \mathbb{N}.
\]

**Proof of Lemma 4.13.** Let \( S > 0 \) and place disjoint balls \( B_S(p_i) \) in the unit ball obtaining a total number of balls proportional to \( 1/S^m \) (consider for instance a grid of length \( 2S \) and put a ball \( B_S(p_i) \) in each cube).
Fix \( k \in \mathbb{N} \). If for a ball we have
\[ B_{S/2}(p_i) \cap \Phi_k(\Sigma) \neq \emptyset, \]
there exists \( q_i \in B_{S/2}(p_i) \) with \( \theta_{k,q_i} \geq 1 \). Since \( B_{S/2}(q_i) \subset B_S(p_i) \), Corollary 4.12 gives
\[
\text{Area} \left( \Phi_k(\Sigma) \cap B_S(p_i) \right) \geq \text{Area} \left( \Phi_k(\Sigma) \cap B_{S/2}(q_i) \right)
\geq \frac{S^2\pi}{6} - \frac{S^2}{8} \int_{B_{S/2}(q_i)} |\tilde{H}|^2 \text{dvol}_{g_{\tilde{\Phi}_k}}.
\]
Since the balls \( B_S(p_i) \) are disjoint and all contained in \( B_1(0) \),
\[
\frac{S^2\pi}{6} \cdot \left| \{ i \text{ s.t. } B_{S/2}(p_i) \cap \Phi_k(\Sigma) \neq \emptyset \} \right|
\leq \text{Area} \left( \Phi_k(\Sigma) \cap B_1(0) \right) + \frac{S^2}{8} \int_{B_1(0)} |\tilde{H}|^2 \text{dvol}_{g_{\tilde{\Phi}_k}}
\leq \left( \frac{3}{2} + \frac{S^2}{8} \right) W(\Phi_k),
\]
where we applied Corollary 4.11 in the last step.

Consequently, due to assumption (4.59), for \( S > 0 \) chosen small enough, there exists for each \( k \in \mathbb{N} \) a point \( p_{i_k} \) such that
\[ B_{S/2}(p_{i_k}) \cap \tilde{\Phi}_k(\Sigma) = \emptyset. \]
(If \( \frac{m}{2^m} \) is the total number of balls \( B_{S/2}(p_i) \) in \( B_1(0) \), choose \( S > 0 \) in such a way that \( \frac{m}{6S^{m-2}} > \left( \frac{3}{2} + \frac{S^2}{8} \right) \sup_k W(\tilde{\Phi}_k). \) Extract a subsequence such that \( p_{i_k} = p_0 \) is independent of \( k \in \mathbb{N} \). \( B_{S/2}(p_0) \) is the ball we have been looking for.

Proof of Theorem 4.3 continued. Recall (1.30) and apply Lemma 4.13 to the sequence \( \tilde{\Psi}_k \) and let \( B_r(p_0) \) be the obtained ball free of mass. Consider the inversion
\[ i_0: x \mapsto \frac{x - p_0}{|x - p_0|^2}, \]
(4.61)
which is a conformal transformation of \( \mathbb{R}^m \cup \{ \infty \} \) such that

\[ \tilde{\Psi}_k(\Sigma) \cap \{ \text{center of inversion of } i_0 \} = \emptyset. \]

Note that \( i_0 \) is a diffeomorphism from \( B_R(p_0) \setminus B_r(p_0) \) into \( B_{1/r}(0) \setminus B_{1/R}(0) \), for any \( R \in (0, \infty) \). Thus,

\[ \| \nabla i_0 \|_{L^\infty(B_R(p_0) \setminus B_r(p_0))} + \| \nabla i_0^{-1} \|_{L^\infty(B_{1/r}(0) \setminus B_{1/R}(0))} \leq C_R. \quad (4.62) \]

Since \( i_0 \) is conformal it satisfies the equation

\[ di_0(x) = e^{\nu(x)} R \]

for \( R \in O(m) \) being some orthogonal matrix. (4.62) implies that for the conformal factor, we have

\[ \| \nu \|_{L^\infty(B_R(p_0) \setminus B_r(p_0))} \leq \tilde{C}_R. \quad (4.63) \]

Define for \( k \in \mathbb{N} \),

\[ \tilde{\Psi}_k := i_0 \circ \tilde{\Psi}_k, \]

and let \( \hat{\alpha}_k = \hat{\alpha}_k + \nu \) denote its conformal factor satisfying \( e^{2\hat{\alpha}_k} h_k = g_{\tilde{\Psi}_k} \).

From (4.58) and the choice of \( B_r(p_0) \), we know that for \( \delta > 0 \) and all \( k \in \mathbb{N} \),

\[ \tilde{\Psi}_k(\Sigma \setminus \bigcup_{i=1, \ldots, n} B_\delta(a_i)) \subset B_{C_\delta}(0) \setminus B_r(p_0). \]

Together with (4.57) and (4.63), this implies that for the conformal factors, we have again

\[ \sup_k \| \hat{\alpha}_k \|_{L^\infty(\Sigma \setminus \bigcup_{i=1, \ldots, n} B_\delta(a_i))} \leq \tilde{C}_\delta. \quad (4.64) \]

What we have gained by inverting is that

\[ \tilde{\Psi}_k(\Sigma) \subset B_{1/r}(0) \quad \text{for all } k \in \mathbb{N}. \quad (4.65) \]

Corollary 4.11 implies that for all \( k \in \mathbb{N} \),

\[ \int_{\Sigma} e^{2\hat{\alpha}_k} dvol_{h_k} = \text{Area}(\tilde{\Psi}_k(\Sigma)) \leq \frac{3}{2r^2} \sup_k W(\tilde{\Phi}_k) \leq C, \quad (4.66) \]

by (1.30).
Step 2d): Weak convergence of $\vec{\Psi}_k$ to $\vec{\xi}_\infty$. Since (4.64) holds for any $\delta > 0$, and due to (4.66), we can argue exactly as in Step 1b) and extract a subsequence such that

$$\vec{\Psi}_k' \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W^{2,2}_{\text{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\}). \quad (4.67)$$

Furthermore, Step 1c) shows that $\vec{\xi}_\infty$ is conformal and we have

$$\log |d\vec{\Psi}_k|^2 \overset{*}{\rightharpoonup} \log |d\vec{\xi}_\infty|^2 \quad \text{in } (L^\infty)^*_\text{loc}(\Sigma \setminus \{a_1, \ldots, a_N\}).$$

It remains to prove Condition iv) from Definition 4.1. Since $h_k \to h_\infty$, (4.66) implies that

$$\sup_k \int_{\Sigma} |d\vec{\Psi}_k|^2 h_\infty \, dvol_{h_\infty} < \infty.$$ 

Together with (4.67), this implies that for any $\delta > 0$,

$$\int_{\Sigma \setminus (\bigcup_{i=1}^n B_\delta(a_i))} |d\vec{\xi}_\infty|^2 h_\infty \, dvol_{h_\infty} \leq \liminf_k \int_{\Sigma} |d\vec{\Psi}_k|^2 h_\infty \, dvol_{h_\infty} \leq C,$$

where $C$ is independent of $\delta$. Thus, $\vec{\xi}_\infty$ extends to a map in $W^{1,2}(\Sigma)$ and we have

$$\vec{\Psi}_k' \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } W^{1,2}(\Sigma).$$

By (4.65),

$$\sup_k \|\vec{\Psi}_k\|_{L^\infty(\Sigma)} < \infty$$

which implies in a similar way

$$\vec{\Psi}_k \rightharpoonup \vec{\xi}_\infty \quad \text{weakly in } (L^\infty)^*(\Sigma).$$

This finishes the proof of Theorem 4.3 for $\vec{\xi}_k := \vec{\Psi}_k$. \hfill \square

5 Weak branched immersions

5.1 Expansion at a blow-up point

Motivated by the compactness result of Theorem 4.3, the purpose of this section is to find out more about the limit object $\vec{\xi}$ of a weakly convergent sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ satisfying $\sup_k \|\vec{\Phi}_k\| < \infty$. The first observation is on the Gauss map $\vec{n}_\xi$. 

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Lemma 5.1. Let $\vec{\xi}$ be the weak limit of a weakly convergent sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ in the sense of Definition 4.1 which satisfies $\sup_k \| \vec{\Phi}_k \| < \infty$. Then

$$\vec{n}_\vec{\xi} \in W^{1,2}(\Sigma).$$

Proof. Denote the blow-up points of $\vec{\xi}$ by $a_1, \ldots, a_N$. If follows from Lemma 4.2 that for any $\delta > 0$, we have

$$\int_{\Sigma \setminus (\cup_{i=1, \ldots, N} B_\delta(a_i))} |d\vec{n}_{\vec{\xi}}|^2_g \, dvol_g \leq \liminf_k \int_{\Sigma} |d\vec{n}_{\vec{\Phi}_k}|^2_{g_k} \, dvol_{g_k} = \liminf_k \mathbb{I}(\vec{\Phi}_k) \leq C.$$

Hence, $\vec{n}_\vec{\xi} \in W^{1,2}_{\text{loc}}(\Sigma \setminus \{a_1, \ldots, a_N\})$ and since $C$ is independent of $\delta$, $\vec{n}_\vec{\xi}$ extends to a map in $W^{1,2}(\Sigma)$. \qed

The following lemma helps to understand the behavior of $\vec{\xi}$ at its blow-up points.

Lemma 5.2. Let $\vec{\xi}: D^2 \to \mathbb{R}^m$ be a weakly conformal map such that $\log |\nabla \vec{\xi}| \in L^\infty(D^2 \setminus \{0\})$ and $\vec{\xi} \in W^{2,2}_{\text{loc}}(D^2 \setminus \{0\})$. Assume $\vec{\xi}$ extends to a map in $W^{1,2}(D^2)$ and that the corresponding Gauss map $\vec{n}_\vec{\xi}$ also extends to a map in $W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$.

Then $\vec{\xi} \in W^{1,\infty}(D^2)$ and there exists $n \in \mathbb{N} \setminus \{0\}$ and a constant $C$ such that

$$(C - o(1)) |z|^{n-1} \leq |\partial_z \vec{\xi}| \leq (C + o(1)) |z|^{n-1}. \quad (5.1)$$

Proof. Problem 8.

Remark 5.3. (5.1) tells us that the behavior of $\vec{\xi}$ at its blow-up point is just the one of a holomorphic curve such as

$$\mathbb{C} \to \mathbb{C}^2, \quad z \mapsto (z^2, z^3).$$

We thus call a blow-up point branch point if it has positive branching order $n - 1 > 0$, where $n \in \mathbb{N} \setminus \{0\}$ is given by Lemma 5.2. (Note that if $n - 1 = 0$, there is no branching and we can remove the singularity.)

Proof of Lemma 5.2. We can localize in order to ensure that

$$\int_{D^2} |\nabla \vec{n}_\vec{\xi}|^2 \, dx \, dy < \frac{8\pi}{3}.$$
Exactly as in Subsection 3.2, using Hélein’s lifting theorem, we deduce the existence of a framing

\( \vec{e} := (\vec{e}_1, \vec{e}_2) \in W^{1,2}(D^2, S^{m-1} \times S^{m-1}) \)

such that

\[
\langle \vec{e}_1, \vec{e}_2 \rangle = 0, \quad \vec{n}_\xi = \star (\vec{e}_1 \wedge \vec{e}_2),
\]

(5.2)

\[
\int_{D^2} \left[ |\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2 \right] \, dx \, dy \leq C \int_{D^2} |\nabla \vec{n}_\xi|^2 \, dx \, dy \quad (5.3)
\]

and satisfying the Coulomb condition

\[
\begin{aligned}
\text{div}(\vec{e}_1, \nabla \vec{e}_2) &= 0 \quad \text{in } D^2 \\
\left( \vec{e}_1, \frac{\partial \vec{e}_2}{\partial \nu} \right) &= 0 \quad \text{on } \partial D^2.
\end{aligned}
\]

(5.4)

We introduce \( e_i := d\vec{\xi}^{-1} \vec{e}_i \) and \( e^*_i \) to be the dual framing. Denoting \( |\partial_x \vec{\xi}|^2 = |\partial_y \vec{\xi}|^2 = e^{2\lambda} \) we have that the metric \( g := \vec{\xi}^* g_{\mathbb{R}^m} \) is given by \( g = e^{2\lambda} [dx^2 + dy^2] \). Hence with respect to the flat metric \( g_0 := [dx^2 + dy^2] \) one has

\[
|e_i|_{g_0}^2 = g_0(e_i, e_i) = e^{-2\lambda} g(e_i, e_i) = e^{-2\lambda}.
\]

and since \( e^*_j(e_i) = \delta_{ij} \) we have that \( |e^*_i|_{g_0}^2 = e^{2\lambda} \). Since \( \vec{\xi} \) is assumed to be in \( W^{1,2}(D^2) \), we deduce that

\[
e_i^* \in L^2(D^2).
\]

Since \( \vec{\xi} \) is in \( W^{1,\infty} \cap W^{2,2}_{\text{loc}}(D^2 \setminus \{0\}, \mathbb{R}^m) \) and \( \log |\nabla \vec{\xi}| \in L^\infty_{\text{loc}}(D^2 \setminus \{0\}) \) we have that the framing given by \( \vec{f}_i := e^{-\lambda} \partial_{x_i} \vec{\xi} \) is in \( L^\infty_{\text{loc}} \cap W^{1,2}_{\text{loc}}(D^2 \setminus \{0\}, \mathbb{R}^m) \).

Since \( \vec{\xi} \) is conformal the unit framing \( (\vec{f}_1, \vec{f}_2) \) is Coulomb:

\[
\text{div}(\vec{f}_1, \nabla \vec{f}_2) = 0 \quad \text{in } D^2 \setminus \{0\}.
\]

Denoting by \( e^{i\theta} \) the rotation which passes from \( (\vec{f}_1, \vec{f}_2) \) to \( (\vec{e}_1, \vec{e}_2) \), the Coulomb condition satisfied by the two framings implies that \( \theta \) is harmonic on \( D^2 \setminus \{0\} \) and hence analytic on this domain. This implies that

\[
e_i^* \in L^\infty_{\text{loc}} \cap W^{1,2}_{\text{loc}}(D^2 \setminus \{0\}).
\]
As in Subsection 3.3 we introduce $f \in W^{1,2}(D^2)$ as the solution to

$$
\begin{cases}
  df = *g(\hat{e}_1, d\hat{e}_2) & \text{on } D^2 \\
  \int_{\partial D^2} f = 0.
\end{cases}
$$

(5.5)

Then $f$ satisfies

$$
\begin{cases}
  \Delta g f = (\nabla^i e_1, \nabla e_2) & \text{on } D^2 \\
  f = 0 & \text{on } \partial D^2
\end{cases}
$$

and Theorem 2.7 implies that $f \in C^0(D^2)$. As in Subsection 3.3, we obtain for $i = 1, 2$

$$
d[e^{-f} e_i^*] = 0 \quad \text{in } \mathcal{D}'(D^2 \setminus \{0\}).
$$

By the Schwartz Lemma the distribution $d[e^{-f} e_i^*]$ is a finite linear combination of successive derivatives of the Dirac Mass at the origin but since $e^{-f} e_i^* \in L^2(D^2)$, this linear combination can only be 0. Hence we have for $i = 1, 2$

$$
d[e^{-f} e_i^*] = 0 \quad \text{in } \mathcal{D}'(D^2).
$$

Hence, by Poincaré’s Lemma, there exists $(\sigma_1, \sigma_2) \in W^{1,2}(D^2, \mathbb{R}^2)$ such that

$$
d\sigma_i = e^{-f} e_i^*.
$$

The dual basis $(\partial/\partial \sigma_1, \partial/\partial \sigma_2) = e^f(e_1, e_2)$ is positive and orthogonal on $D^2 \setminus \{0\}$. Hence $\sigma = \sigma_1 + i\sigma_2$ is an holomorphic function on $D^2 \setminus \{0\}$ which extends to a $W^{1,2}$-map on $D^2$. The classical point removability theorem for holomorphic maps implies that $\sigma$ extends to an holomorphic function on $D^2$. Possibly after modifying $\sigma$ by a constant, we can assume that $\sigma(0) = 0$. The holomorphicity of $\sigma$ implies in particular that

$$
|d\sigma|_{g_0} = \sqrt{2} e^{\lambda - f}
$$

is uniformly bounded and, since $f \in L^\infty(D^2)$, we deduce that $\lambda$ is bounded from above on $D^2$. This fact implies that $\xi$ extends to a Lipschitz map on $D^2$. Though $|d\sigma|_{g_0}$ has no zero on $D^2$, $\sigma'$ might have a zero at the origin: there exists an holomorphic function $h(z)$ on $D^2$ satisfying $h(0) = 0$, a complex number $c_0$ and an integer $n$ such that

$$
\sigma(z) = c_0 \ z^n (1 + h(z)).
$$

(5.6)
We have that locally
\[ \partial_\sigma \tilde{\xi} = \partial_{\sigma_1} \tilde{\xi} - i\partial_{\sigma_2} \tilde{\xi} = d\tilde{\xi}(e^f e_1) - i d\tilde{\xi}(e^f e_2) = e^f [e_1 - ie_2]. \]

Hence, since \( f \) is continuous, we have that
\[ |\partial_\sigma \tilde{\xi}| = \sqrt{2} e^{f(0)} (1 + o(1)). \] (5.7)

Combining (5.6) and (5.7) gives
\[ |\partial_z \tilde{\xi}| = |\partial_\sigma \tilde{\xi}| |\partial_z \sigma| = c_0 n \sqrt{2} e^{f(0)} |z|^{n-1} (1 + o(1)). \] (5.8)

This last identity implies (5.1).

\[ \square \]

**Definition 5.4.** Let \((\Sigma, h)\) be a conformal structure on \( \Sigma \), where \( h \) denotes the associated metric of constant curvature and unit volume. The space \( F_{\text{conf}}(\Sigma, h) \) denotes the set of measurable maps \( \tilde{\xi}: \Sigma \to \mathbb{R}^m \) that satisfy

i) \( \tilde{\xi} \in W^{1, \infty}(\Sigma) \);

ii) \( \tilde{\xi}: (\Sigma, h) \to \mathbb{R}^m \) is weakly conformal;

iii) there exist finitely many blow-up points \( a_1, \ldots, a_N \in \Sigma \) s.t.
\[ \log |d\tilde{\xi}| \in L^\infty(\Sigma \setminus \{a_1, \ldots, a_N\}); \]

iv) \( n_\tilde{\xi} \in W^{1, 2}(\Sigma, Gr_{m-2}(\mathbb{R}^m)) \).

**Remark 5.5.** Let \( \tilde{\Phi}_k \) be a sequence in \( E_\Sigma \) with \( \sup_k \|\tilde{\Phi}_k\| < \infty \). Let \( h_k \) denote the respective metrics of constant curvature and unit volume of the induced conformal structures, which are assumed to satisfy condition (CA) with \( h_k \to h_\infty \). Suppose \( \tilde{\Phi}_k \) weakly converges to \( \tilde{\xi}_\infty \) in the sense of Definition 4.1. Then Lemma 5.1 and Lemma 5.2 imply that \( \tilde{\xi}_\infty \) is an element of \( F_{\text{conf}}(\Sigma, h_\infty) \).

We are now ready to introduce the space of weak branched immersions, which contains the closure of \( E_\Sigma \) under weak convergence.

**Definition 5.6.** Define the space \( F_\Sigma \) of weak branched immersions as the space of measurable maps \( \tilde{\Phi}: \Sigma \to \mathbb{R}^m \) such that there exists a bi-Lipschitz diffeomorphism \( \Psi \) of \( \Sigma \) and a conformal structure on \( \Sigma \), with \( h \) being the associated constant curvature metric of unit volume, such that \( \tilde{\Phi} \circ \Psi \in F_{\text{conf}}(\Sigma, h) \).
Let $\xi \in \mathcal{F}^{\text{conf}}_{(\Sigma, h)}$ be a weak branched conformal immersion with branch points $\{b_j\}$ and respective branching orders $\{n_j - 1\}$, given by Lemma 5.2. Taking isothermal coordinates around $b_j$, Lemma 5.2 gives us information on the behavior of the conformal factor

$$\lambda = \log |\partial_{x_1} \xi| = \log |\partial_{x_2} \xi| = \log |\partial_z \xi| - \log \sqrt{2}$$

at $0 = \psi^{-1}(b_j)$. More specifically, we have

$$-\Delta \lambda = e^{2\lambda} K \quad \text{in} \quad D^2 \setminus \{0\}$$

where we used Lemma 1.6 on the regular part of $\xi$.

$$-\Delta \lambda = e^{2\lambda} K - 2\pi (n_j - 1) \delta_{b_j}.$$  \hspace{1cm} (5.11)

In the same way, Lemmas 1.7 and 5.2 imply the following extension of Liouville’s equation to maps in $\mathcal{F}^{\text{conf}}_{\Sigma}$.

**Lemma 5.7.** Let $\xi \in \mathcal{F}^{\text{conf}}_{(\Sigma, h)}$ have the branch points $b_1, \ldots, b_N$ with respective branching orders $n_1 - 1, \ldots, n_N - 1 \in \mathbb{N} \setminus \{0\}$, given by Lemma 5.2 and let

$$g = e^{2\alpha} h.$$   \hspace{1cm} (5.12)

Then $\alpha$ satisfies the following PDE in $\mathcal{D}'(\Sigma)$:

$$-\Delta_h \alpha = e^{2\alpha} K_g - K_h - 2\pi \sum_{j=1}^{N} (n_j - 1) \delta_{b_j}.$$   \hspace{1cm} (5.12)

The following lemma gives a control of the branch points with multiplicity.
Lemma 5.8. Let \( \xi \in \mathcal{F}^{\text{conf}}_{(\Sigma,h)} \) have the branch points \( b_1, \ldots, b_N \) with respective branching orders \( n_1 - 1, \ldots, n_N - 1 \in \mathbb{N} \setminus \{0\} \), given by Lemma 5.2 and let \( g = e^{2\alpha} h \).

Then
\[
\sum_{j=1}^{N} (n_j - 1) \leq \frac{1}{4\pi} \mathcal{II}(\xi) - \chi(\Sigma). \tag{5.13}
\]

Proof of Lemma 5.8. Applying the identity (5.12) of distributions to the constant function 1 on \( \Sigma \) yields
\[
2\pi \sum_{j=1}^{N} (n_j - 1) = \int_{\Sigma} e^{2\alpha} K_g \, dv_{1} - \int_{\Sigma} K_h \, dv_{1} \]
\[
= \int_{\Sigma} K_g \, dv_{1} - 2\pi \chi(\Sigma) \leq \frac{1}{2} \mathcal{II}(\xi) - 2\pi \chi(\Sigma),
\]
where we used (1.23) in the last step. \( \square \)

5.2 Weak sequentially closedness of \( \mathcal{F}_\Sigma \)

We called Theorem 4.3 a weak “almost-closure theorem” because starting from a sequence \( \Phi_k \) in \( \mathcal{E}_\Sigma \) the weak limit map \( \xi_\infty \) is in general not contained in the class \( \mathcal{E}_\Sigma \), but only in the strictly larger space \( \mathcal{F}_\Sigma \). In the following theorem we show that the space \( \mathcal{F}_\Sigma \) of weak branched immersions is in fact closed under weak convergence, i.e. we obtain a weak closure theorem.

Theorem 5.9 (Weak closure theorem). Let \( \Phi_k \in \mathcal{F}_\Sigma \) be a sequence such that
\[
\sup_{k} \mathcal{II}(\Phi_k) < \infty. \tag{5.14}
\]

Suppose assumption \( \text{(CA)} \) is satisfied and thus, up to subsequences, for the constant curvature metrics \( h_k \) of unit volume of the conformal structures induced by \( \Phi_k \), we have
\[
h_k \to h_\infty \quad \text{in } C^l(\Sigma), \quad \text{for all } l \in \mathbb{N},
\]
for \( h_\infty \) being the constant curvature metric of unit volume of some conformal structure on \( \Sigma \).

Then there exists a subsequence of \( \Phi_k \) which, in the sense of Definition 4.1, weakly converges to an element of the space \( \mathcal{F}^{\text{conf}}_{(\Sigma,h_\infty)} \subset \mathcal{F}_\Sigma \).
Proof of Theorem 5.9. Compose with diffeomorphisms $\Psi_k$ to obtain $\tilde{\xi}_k := \Phi_k \circ \Psi_k \in F_{\text{conf}}^{(\Sigma, h_k)}$. Denote the branch points of $\xi_k$ by $b^k_1, \ldots, b^k_{N_k}$ with respective branching orders $n^k_1 - 1, \ldots, n^k_{N_k} - 1 \in \mathbb{N} \setminus \{0\}$. (5.12) and (5.13) imply that
\[ \| \Delta \alpha_k \|_{M(\Sigma)} \leq 2 \left( \frac{1}{2} \Pi(\Phi_k) - 2 \pi \chi(\Sigma) \right) \leq C. \]
Consequently, Theorem 2.6 implies that, as in the unbranched case, we get a global bound
\[ \sup_k \| d\alpha_k \|_{L^2, \infty(\Sigma)} < \infty. \]
Thus, Theorem 4.5 holds true for a sequence of isothermal charts $\varphi_k$ satisfying
\[ \sup_k \int_{D^2} |\nabla n_{\tilde{\xi}_k \circ \varphi_k}|^2 dx_1 dx_2 < \frac{8\pi}{3} \] (5.15)
and containing none of the branch points $b^k_1, \ldots, b^k_{N_k}$.

Lemma 5.8 and condition (5.14) imply that the number of branch points is uniformly bounded:
\[ N_k \leq \sum_{j=1}^{N_k} (n^k_j - 1) \leq \frac{1}{4\pi} \Pi(\Phi_k) - \chi(\Sigma) \leq C. \]
Hence, we can extract a subsequence of $\tilde{\xi}_k$ such that $N_{k'} := N_0$ is independent of $k'$.

We perform Step 2a) in the proof of Theorem 4.3 with the only difference that when extracting a further subsequence in order to obtain (4.52) and (4.53), we do this in such a way that additionally, for each $j \in \{1, \ldots, N_0\}$ there exists $b^\infty_j \in \Sigma$ with
\[ b^k_j \to b^\infty_j, \] (5.16)
as $k' \to \infty$. Define
\[ \{d_1, \ldots, d_M\} := \{b^\infty_1, \ldots, b^\infty_{N_0}\} \cup \{a_1, \ldots, a_N\}, \] (5.17)
where the latter set of points is as defined in (4.54).

In Step 2b), if $B_{s_0}^{h_{x_0}}(x_0^\infty)$ contains any of the points $b^\infty_1, \ldots, b^\infty_{N_0}$, choose a smaller ball $\hat{B}_c \subset B_{s_0}^{h_{x_0}}(x_0^\infty)$ that is free of these points.
Let $0 < \delta < \min\{\min_{i \in I_0} \rho_{i,\infty}, t\}$ be arbitrary small. Step 1a) can now be applied to $K = \Sigma \setminus \bigcup_{i=1}^{M} B_{\delta}(d_i)$, with

$$\rho_{\infty} = \min \left\{ \inf_{x \in K} \rho_{x,\infty} \cdot \frac{\delta}{2} \right\}.$$ 

The choice of $\rho_{\infty}$ makes sure that Theorem 4.5 can be applied, i.e. we have a cover of balls satisfying (5.15) and containing no branch points.

The rest of Step 2b) and Steps 2c) and d) imply weak convergence to some $\tilde{\xi}_{\infty} \in \mathcal{F}_{\Sigma,h_{\infty}}^{\text{conf}}$ with blow-up points $d_1, \ldots, d_M$. \hfill \Box
References
