Lecture 2

Kähler geometry. Recall: $\mathbb{C}^n$ complex manifold

$g$ Kähler, $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^\bar{j}$

We have $d\omega = 0$.

If $X = X^i \frac{\partial}{\partial z^i}$ is a $T^1,0$ vector field,

define $\nabla_k X^i = \partial_k X^i + T^i_{kp} X^p$

where $T^i_{kp} = g^{i\bar{q}} \partial_k g_{p\bar{q}}$

And $\nabla_\bar{k} X^i = \partial_\bar{k} X^i$.

Curvature is defined by $R_{ijkp} = -\partial_j T_{ikp}$ (Exercise: this is a tensor).

We will prove:

Prop. $[\nabla_i, \nabla_j] X^k = R_{ijkp} k X^p$

We use the lemma:

Lemma Fix $x \in \mathbb{C}^n$. $\exists z^1, \ldots, z^n$ complex coords centered at $x$ such that $g_{i\bar{j}}|_x = \delta_{i\bar{j}}$

$\partial_k g_{i\bar{j}}|_x = 0 \quad \forall i, j, k$

(in particular, $T^k_{ij}|_x = 0$)

Proof: Problem set 2. $\square$
Proof of Prop:
Since both sides are tensors, it suffices to prove using the coordinates of the lemma, at a fixed point $x$. At $x$, we compute
\[
[\nabla_i, \nabla_j] X^k = \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k
= \partial_i \partial_j X^k - \partial_j (\partial_i X^k + T_{ip}^k X^p) - (\partial_j T_{ip}^k) X^p
= R_{ijp}^k X^p.
\]
\[\square\]

**Kähler-Ricci Flow**

First define Ricci curvature
\[
R_{ij} = g^{k\bar{k}} R_{ijk\bar{k}}
\]

**Lemma**
\[
R_{ij} = -\partial_i \partial_j \log \det g
\]

**Pf Exercise. $\square$**

Define the Ricci form
\[
\text{Ric}(\omega) = \sqrt{-1} R_{ij} dz^i \wedge \bar{dz}^j
\]
\[
= -\sqrt{-1} \partial \bar{\partial} \log \det g
\]

Note that $\text{Ric}(\omega)$ is a d-closed real (1,1) - form.

**Defn:** If $\omega(t)$ is a smooth family of Kähler metrics satisfying
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) \\
\omega|_{t=0} = \omega_0, \quad \omega_0 \text{ fixed Kähler}
\end{array} \right. \quad \text{(KRF)}
\]
we say $\omega(t)$ is a solution of the **Kähler-Ricci Flow**

starting at $\omega_0$. 

\[\text{(KRF)}\]
Maximal existence time

We first talk about cohomology. \( \omega \) Kähler is a closed real \((1,1)\) form, so defines an element \([\omega]\) of \( H^{1,1}_\delta (M; \mathbb{R}) \equiv \text{\# of \( \overline{\partial} \)-closed real \((1,1)\) forms} \) \( \text{Im} \overline{\partial} \).

The \( \overline{\partial} \)-Lemma says that on a Kähler manifold

\[
H^{1,1}_\delta (M; \mathbb{R}) = \text{\# \( \overline{\partial} \)-closed real \((1,1)\) forms} \quad \text{Im} \overline{\partial} 
\]

Hence, if \( \beta \) is a closed real \((1,1)\) form then

\[
[\beta] = [\omega] \iff \beta = \omega + \sqrt{-1} \overline{\partial} \phi \quad \text{for some smooth } \phi : M \to \mathbb{R}.
\]

We say that a class \( \alpha \) is Kähler if \( \exists \omega \) Kähler with \([\omega] = \alpha \), and we write \( \alpha > 0 \).

Define Kähler cone

\[
\text{Ka}(M) = \{ \alpha \in H^{1,1}_\delta (M; \mathbb{R}) \mid \alpha > 0 \}
\]

This is a cone in the finite dimensional real vector space \( H^{1,1}_\delta (M; \mathbb{R}) \). Namely

\[
\alpha, \beta \in \text{Ka}(M) \implies \alpha + \beta \in \text{Ka}(M) \\
\alpha \in \text{Ka}(M), \ t > 0 \implies t\alpha \in \text{Ka}(M)
\]

(Pf: exercise).

We define the first Chern class

\[
c_1(M) = [\text{Ric}(\omega)] \in H^{1,1}(M; \mathbb{R})
\]

for \( \omega \) any Kähler metric.
Prop. $c_1(M)$ is well-defined (independent of choice of $\omega$).

Proof. Let $\omega' = \sqrt{-1} g' j \overline{dz^1} \wedge \cdots \wedge \overline{dz^n}$ be any other Kähler metric. Then

$$\det g' = \det g \cdot e^F$$

for some smooth function $F: M \to \mathbb{R}$.

(Indeed: for $\omega$ Kähler, $\det g$ defines a vol form since $\omega^n = n! (\sqrt{-1})^n \det g dz^1 \wedge \cdots \wedge dz^n \wedge \overline{dz^n}$ is a top-dimensional nowhere vanishing form.)

Then

$$\text{Ric}(\omega') = -\sqrt{-1} \partial \overline{\partial} \log \det g'$$
$$= -\sqrt{-1} \partial \overline{\partial} \log (\det g \cdot e^F)$$
$$= \text{Ric}(\omega) - \sqrt{-1} \partial \overline{\partial} F$$
$$\Rightarrow \mathbf{[\text{Ric}(\omega')] = [\text{Ric}(\omega)]}$$

Hence $c_1(M)$ is an invariant of the complex manifold $M$.

Now return to the Kähler-Ricci flow

\[
\left\{ \begin{array}{l}
\frac{2}{\Delta t} \omega = -\text{Ric}(\omega) \\
\omega|_{t=0} = \omega_0
\end{array} \right. 
\]

Take cohomology classes

\[
\left\{ \begin{array}{l}
\frac{d}{dt} [\omega] = -c_1(M) \\
[\omega]|_{t=0} = [\omega_0]
\end{array} \right. 
\]

the solution of this ODE is

$$[\omega(t)] = [\omega_0] - t c_1(M).$$

Hence, as long as the flow exists we have

$$[\omega_0] - t c_1(M) > 0 \quad \text{(since it contains $\omega(t)$ which is Kähler.)}$$
The maximal existence time theorem says that this necessary condition is sufficient for existence of a solution to the flow:

**Theorem (H.-D. Cao, Tsuji, Tian-Zhang)** Let maximal solution \( w(t) \) to KRF starting at \( w_0 \) for \( t \in [0,T) \) where
\[
T = \sup \{ t > 0 \mid [w_0] - tc_1(M) > 0 \}
\]
Namely, the flow exists as long as the straight line path \( t \mapsto [w_0] - tc_1(M) \) remains in the Kähler cone.

There are many possibilities:

1. \( K_a(M) \) tends to zero
   - Only happens if \( c_1(M) > 0 \)

2. Kähler class doesn't move
   - \( c_1(M) = 0 \)

3. Stays in cone
   - Flow exists for all time
   - Occurs if, for example, \( c_1(M) < 0 \)

4. Hits a non-zero element of boundary of \( K_a(M) \).
Examples.

1) Let $M$ be a Riemann surface ($n=1$).

By Uniformization Theorem, $\omega_{KE}$ Kähler-Einstein metric with $\text{Ric}(\omega_{KE}) = \mu \omega_{KE}$ for $\mu = 1, 0, -1$.

- $\mu = 1$

\[ S^2 = \mathbb{P}^1 \]

$\omega_{KE}$ = Fubini-Study metric
Round metric

- $\mu = 0$

Torus $T^2$
Flat metric

- $\mu = -1$

Surface genus $> 1$
Hyperbolic metric

On $\mathbb{P}^1$, if we start with $\omega_0 = \omega_{KE}$, we have a soln to KRF $\omega(t) = (1-t)\omega_{KE}$.

Indeed $\frac{\partial}{\partial t} \omega(t) = -\omega_{KE}$

$\quad = -\text{Ric}(\omega_{KE})$

$\quad = -\text{Ric}(\omega(t))$

where we use fact that for $\lambda > 0$, $\text{Ric}(\lambda \omega) = \lambda \text{Ric}(\omega)$ by defn of $\text{Ric}(\omega)$.

On $T^2$ we have stationary soln $\omega(t) = \omega_{KE}$

On surface of genus $> 1$, $\omega(t) = (1+t)\omega_{KE}$ expanding solution exists for all time.

$K_a(M)$

$\mu = 1$

$\leftarrow$

$\mu = 0$

$\rightarrow$

$\mu = -1$
Theorem (Hamilton, Chow) Assume \( n = 1 \), \([\omega_0] = [\omega_{KE}]\).

Then KRF starting at \( \omega_0 \) has "same limiting behavior" as when \( \omega_0 = \omega_{KE} \).

E.g. If \( M = T^2 \) then KRF exists for all time and converges to flat metric.

If \( M = \mathbb{S}^1 \) the flow exists on \([0,1]\) and "converges to a round point" (modulo automorphisms). 

2) \( \mathbb{S}^1 \times \mathbb{S}^1 \), \( \pi_1, \pi_2 \) projections, \( \omega_{KE} \) on \( \mathbb{S}^1 \).

Let \( \beta_1 = \pi_1^* \omega_{KE}, \beta_2 = \pi_2^* \omega_{KE} \).

Then \( \beta_1 + \beta_2 \) is product metric.

\( \text{Ric}(\beta_1 + \beta_2) = \beta_1 + \beta_2 \) (see problem set 2).

Hence \( c_1(M) = [\beta_1] + [\beta_2] \).

\([\beta_1], [\beta_2] \) span \( H^1(M; \mathbb{R}) \).

There are 3 behaviors of flow depending on where \([\omega_0] \) lies.

(assume \( \omega_0 \) is a product of metrics on \( \mathbb{S}^1 \)).

(i) \([\omega_0]\) on diagonal \( \omega(t) \) shrinks to a point

(ii) \([\omega_0]\) below diagonal \( \omega(t) \) collapses to first \( \mathbb{S}^1 \).
[W₀] above diagonal
\[\omega(t)\] collapse to second \(D'.\)