1. Problem Set 1

(1) Let $\alpha = \phi_1 dx + \phi_2 dy + \phi_3 dz$, where $\phi_1, \phi_2, \phi_3 \in \mathbb{C}[x, y, z]$, be a polynomial 1-form on $\mathbb{C}^3$. Define a skew-symmetric bracket $\{-, -\}_\alpha : \mathbb{C}[x, y, z] \times \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z]$, by the formula

$$\{-, -\}_\alpha (f, g) = \frac{\alpha \wedge df \wedge dg}{dx \wedge dy \wedge dz}. \quad (1.0.1)$$

(Here the 3-form in the numerator is necessarily of the form $h dx \wedge dy \wedge dz$ for some polynomial $h$, and the fraction stands for that $h$).

- Show that the Jacobi identity holds for $\{-, -\}_\alpha$ iff the $\alpha \wedge d\alpha = 0$. (More generally, let $X$ be a smooth variety, $\text{vol} \in \text{Poly}^3(X)$ a nowhere vanishing 3-polyvector, and $\alpha$ a 1-form. Then, the bivector $\Pi = i_\alpha \text{vol}$ is Poisson iff $\alpha \wedge d\alpha = 0$.)
- Take $\alpha := d\phi$ for some nonzero polynomial $\phi$. Thus, $\{-, -\}_d\phi$ is a nonzero Poisson bracket on $\mathbb{C}[x, y, z]$. Find $\{x, y\}_d\phi$.
- Show that $\mathbb{C}[\phi]$, the subalgebra generated by $\phi$, is contained in the Poisson center of $\mathbb{C}[x, y, z]$.
- Show that any sufficiently general level set of $\phi$ is a symplectic leaf.
- Deduce that the Poisson center equals

$$\{f \in \mathbb{C}[x, y, z] \mid f \text{ is algebraic over } \mathbb{C}[\phi]\}.$$  

The Poisson bracket $\{-, -\}_d\phi$ descends to the quotient $A_\phi := \mathbb{C}[x, y, z]/(\phi)$. Classify all symplectic leaves in $\text{Spec} A_\phi$ in the case where $\phi$ has isolated critical points in $\mathbb{C}^3$.

(2) Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$.

- Show that symplectic leaves in $\mathfrak{g}^*$ are precisely the coadjoint $G$-orbits.
- Let $\lambda \in \mathfrak{g}^*$ and $x, y \in \mathfrak{g}$. The vectors $u = \text{ad}^* x(\lambda), v = \text{ad}^* y(\lambda)$, where $\text{ad}^*$ denotes the (co)adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, are tangent to the $G$-orbit $O$ of the element $\lambda$. Find $\omega(u, v)$, the value of the canonical symplectic 2-form $\omega$ on the leaf $O$ at the vectors $(u, v)$.

(3) Let $G$ be a Lie group and $H \subset G$ a Lie subgroup. Let $\mathfrak{g} = \text{Lie} G$, resp. $\mathfrak{h} = \text{Lie} H$, and identify $(\mathfrak{g}/\mathfrak{h})^*$ with $\mathfrak{h}^\perp \subset \mathfrak{g}^*$, the annihilator of the subspace $\mathfrak{h} \subset \mathfrak{g}$. We have natural identifications of vector bundles

$$T^*(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})^* \cong G \times_H \mathfrak{h}^\perp. \quad (\dagger)$$

- Let $\lambda, \alpha, \beta \in \mathfrak{h}^\perp$ and $x, y \in \mathfrak{g}$. We view $\lambda$ as an element of the fiber of $T^*(G/H)$ over the base point $1/H$, and $\alpha, \beta$ as ‘vertical tangent vectors’ at $\lambda \in T^*(G/H)$, i.e. as elements of the tangent space $T_\lambda(T^*(G/H))$ which are tangent to the fiber of the projection $T^*(G/H) \to G/H$. Similarly, write $x(\lambda), y(\lambda)$ for the elements of $T_\lambda(T^*(G/H))$ tangent to the $G$-orbit of $\lambda$ under the $G$-action on $T^*(G/H)$.

Let $\omega$ be the canonical symplectic 2-form on $T^*(G/H)$. Express each of the numbers:

$$\omega(\alpha, \beta), \omega(x(\lambda), y(\lambda)), \omega(x(\lambda), \alpha)$$

in terms of $\lambda, \alpha, \beta$ and $x, y$.

- The group $G$ acts on $G/H$ by left translations, so we have a Hamiltonian action of $G$ on $T^*(G/H)$. Give a formula for the corresponding moment map, viewed as a map $G \times_H \mathfrak{h}^\perp \to \mathfrak{g}^*$.

(4) Let $(V, \omega)$ be a (finite dimensional) symplectic vector space and $\Gamma \subset Sp(V, \omega)$ a finite subgroup.

- Classify all symplectic leaves in $V/\Gamma$.
- More difficult: Let $U$ be the unique open dense symplectic leaf in $V/\Gamma$ and $\omega$ the corresponding symplectic 2-form on $U$. Prove that for any resolution of singularities
\( \pi : X \to V/\Gamma \) the 2-form \( \pi^* \omega_i \), on \( \pi^{-1}(U) \), extends to a regular (possibly degenerate) 2-form on the whole of \( X \).

Hint: Given a resolution of singularities \( \pi : X \to V/\Gamma \), consider a resolution of singularities of the variety \( (X \times_{V/\Gamma} V)_{\text{red}} \).

2. Problem set 2

(1) Let \( \langle e, h, f \rangle \) be the standard basis of the Lie algebra \( \mathfrak{g} = \mathfrak{sl}_2 \). We identify \( \mathbb{C}[\mathfrak{g}^*] \) with \( \mathbb{C}[e, h, f] \).
- Write an explicit formula for the Poisson bracket on \( \mathbb{C}[e, h, f] \) transported from the canonical one on \( \mathbb{C}[\mathfrak{g}^*] \).
- Show that the Poisson center of the algebra \( \mathbb{C}[e, h, f] \) is generated by the polynomial \( P = h^2 + 2ef \).

(2) Find explicit formulas for each of the maps \( \mu \) defined below:
- Let \( (V, \omega) \) be a symplectic vector space. The natural action of \( Sp(V, \omega) \), the symplectic group, on \( V \) is Hamiltonian. Identify \( (\text{Lie } Sp(V, \omega))^* \) with \( \text{Lie } Sp(V, \omega) \) via the trace pairing. Let \( \mu : V \to \text{Lie } Sp(V, \omega) \) be the map obtained, using this identification, from the moment map \( V \to (\text{Lie } Sp(V, \omega))^* \).
- Let \( G \), a Lie group, act on its Lie algebra \( \mathfrak{g} \) by the adjoint action. This gives a Hamiltonian action of \( G \) on \( T^* \mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g} \) with moment map \( T^* \mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}^* \). Given an invariant nondegenerate symmetric bilinear form \( \langle -, - \rangle \) on \( \mathfrak{g} \) let \( \mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) be the map obtained from the moment map via the identifications \( \mathfrak{g}^* \cong \mathfrak{g}, \) resp. \( T^* \mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g} \).
- Let \( \text{Rep}(Q, \mathfrak{d}) \) be the variety of \( \mathfrak{d} \)-dimensional representations of a quiver \( Q \). The group \( \text{G}_\mathfrak{d} \) acts on \( \text{Rep}(Q, \mathfrak{d}) \). Let \( \mu : \text{Rep}(Q, \mathfrak{d}) \to \mathfrak{g}_\mathfrak{d} \) be the map obtained from the moment map \( T^* \text{Rep}(Q, \mathfrak{d}) \to (\mathfrak{g}_\mathfrak{d})^* \) via the identifications \( T^* \text{Rep}(Q, \mathfrak{d}) \cong \text{Rep}(Q, \mathfrak{d}) \), resp. \((\mathfrak{g}_\mathfrak{d})^* \cong \mathfrak{g}_\mathfrak{d} \).

(3) Fix \( n \geq 2 \) and let \( \Gamma_n \cong \mathbb{Z}/(n) \) be the group of \( n \)-th roots of unity. We have an imbedding \( \Gamma_n \hookrightarrow SL_2(\mathbb{C}), \zeta \mapsto \text{diag}(\zeta, \zeta^{-1}) \). Since \( SL_2(\mathbb{C}) = Sp(\mathbb{C}^2) \), this gives a \( \Gamma_n \)-action on \( \mathbb{C}^2 \) that preserves the standard symplectic form. Hence, the induced \( \Gamma_n \)-action on the polynomial algebra \( \mathbb{C}[u,v] = \mathbb{C}[\mathbb{C}^2] \), respects the Poisson bracket that comes from the symplectic form. Thus, the algebra \( \mathbb{C}[u,v]^{\Gamma_n} \) is a Poisson subalgebra of \( \mathbb{C}[u,v] \).

Construct an isomorphism of Poisson algebras
\[
\mathbb{C}[u,v]^{\Gamma_n} \cong A_{\phi_n}, \quad \text{where} \quad \phi_n := x^2 + y^2 + z^n.
\]
(we’ve used the notation of Problem 1 from Problem Set 1.)

3. Problem set 3

(1) Let \( V \) be a finite dimensional vector space. The group \( GL(V) \) acts naturally on \( V \) and it also acts on \( \text{gl}(V) = \text{Lie } GL(V) \) by conjugation. We let \( GL(V) \) act diagonally on the vector space \( \text{gl}(V) \oplus V \). This gives the Hamiltonian \( GL(V) \)-action on
\[
T^*(\text{gl}(V) \oplus V) = \text{gl}(V) \oplus \text{gl}(V) \oplus V \oplus V^*.
\]
We will write an element of the cotangent space as a quadruple \( (x,y,i,j) \) where \( x, y \in \text{gl}(V), \quad i, j \in V^* \).
- Find an explicit formula for the moment map
\[
\mu : \text{gl}(V) \oplus \text{gl}(V) \oplus V \oplus V^* \to \text{gl}(V) \cong \text{gl}(V)^*.\]
- Show that the \( GL(V) \)-action on \( \mu^{-1}(\text{Id}) \), the fiber of \( \mu \) over the identity \( \text{Id} \in \text{gl}(V) \), is free. So, \( M := \text{Spec}(\mathbb{C}[\mu^{-1}(\text{Id})]^{GL(V)}) \), the corresponding Hamiltonian reduction, is a smooth symplectic affine variety.
- Find \( \text{dim } M \).
• Consider a collection of functions on the vector space (†) given by the formulas
  \[ a_n(x, y, i, j) = \text{Tr}(x^n), \quad b_n(x, y, i, j) = \text{Tr}(y^n), \quad n = 1, 2, \ldots. \]
  These functions are \( GL(V) \)-invariant, hence they descend to regular functions \( \bar{a}_n, \bar{b}_n \in \mathbb{C}[M] \).
  Show that
  \[ \{ \bar{a}_n, \bar{a}_m \} = \{ \bar{b}_n, \bar{b}_m \} = 0 \quad \forall m, n \geq 1. \]
  and find \( \{ \bar{a}_n, \bar{b}_m \} \) for all \( 1 \leq m, n \leq 2 \).

• Use the First Fundamental Theorem of Invariant Theory to prove that if \( \dim V = 2 \) then the functions \( \bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2 \) generate \( \mathbb{C}[M] \) as a Poisson algebra, i.e. that the only Poisson subalgebra of \( \mathbb{C}[M] \) that contains all four functions \( \bar{a}_n, \bar{b}_n, n = 1, 2 \), is \( \mathbb{C}[M] \) itself. (These functions do not generate \( \mathbb{C}[M] \) as a commutative algebra !).

(1) Let \( W = \mathbb{Z}/(2) = \{ 1, s \} \) be the Weyl group of the root system \( A_1 \) and \( e := \frac{1}{2}(1 + s) \), an idempotent in the group algebra. Let \( H_{t,c}(A_1) \) be the corresponding symplectic reflection algebra with parameters \( t, c \in \mathbb{C} \) and \( e H_{t,c}(A_1) e \) its spherical subalgebra.

• Show that, for \( t \neq 0 \), the algebra \( e H_{t,c}(A_1) e \) is generated by the elements \( e x^2 \) and \( e y^2 \).

• Establish an algebra isomorphism
  \[ e H_{1,c}(A_1) e \cong \mathcal{U}g / (\Delta), \]
  for an appropriate central element \( \Delta \in \mathcal{U}g \).

• Show that \( e H_{0,c}(A_1) e \) is a commutative algebra and, moreover, the assignment \( e x^2 \mapsto x, \ e y^2 \mapsto y \), extends uniquely to a Poisson algebra isomorphism
  \[ e H_{0,c}(A_1) e \overset{\sim}{\rightarrow} A_{\phi}, \]
  where \( \phi = x^2 + y^2 + z^2 - \frac{c(c+1)}{2} \) (notation of Problem 1 from Problem Set 1).

(2) Let \( W = \mathbb{Z}/(2) = \{ 1, s \} \) be the Weyl group of the root system \( A_1 \) and \( e := \frac{1}{2}(1 + s) \), an idempotent in the group algebra. Let \( H_{t,c}(A_1) \) be the corresponding symplectic reflection algebra with parameters \( t, c \in \mathbb{C} \) and \( e H_{t,c}(A_1) e \) its spherical subalgebra.

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  \[ e H_{0,c}(A_1) e \overset{\sim}{\rightarrow} A_{\phi}, \]
  where \( \phi = x^2 + y^2 + z^2 - \frac{c(c+1)}{2} \) (notation of Problem 1 from Problem Set 1).

(3) Let \( G \) be a connected semisimple group with trivial center, and \( (e, h, f) \) an \( sl_2 \)-triple in \( g \), the Lie algebra of \( g \). The \( ad \)-\( h \)-action on \( g \) is semisimple with integer eigenvalues, hence it gives a \( \mathbb{Z} \)-grading \( g = \bigoplus_{i \in \mathbb{Z}} g_i \). Put \( p = \bigoplus_{i \geq 0} g_i \). This is a parabolic subalgebra of \( g \). Let \( P \) be the corresponding parabolic subgroup of \( G \). The subspace \( g_{\geq 2} := \bigoplus_{i \geq 2} g_i \) of \( g \) is \( Ad P \)-stable, so we define
  \[ X := G \times_{\nu} g_{\geq 2}. \]

The group \( G \) acts on \( X \) by \( g : (h, x) \mapsto (gh, x) \) for all \( g, h \in G, x \in g_{\geq 2} \). Further, the assignment \( (h, x) \mapsto Ad h(x) \) gives a \( G \)-equivariant map \( \pi : X \rightarrow g \).

• Check that \( g_{\geq 2} \) is an \( Ad P \)-stable subspace of \( \mathfrak{p} \) and the \( P \)-orbit of the element \( e \in g_{\geq 2} \)
  is Zariski open and dense in \( g_{\geq 2} \). [Hint: Use representation theory of \( sl_2 \) to prove that the tangent space to this orbit equals \( g_{\geq 2} \).]

• Show that \( \pi \) is proper and its image equals \( \overline{Ad G(e)} \), the closure of the \( G \)-orbit of \( e \) in \( g \).

• Show that \( \pi \) restricts to an isomorphism \( \overline{Ad G(e)} \cong Ad G(e) \), hence, it is a birational isomorphism of \( X \) and \( \overline{Ad G(e)} \).

• Identify \( g^* \cong g \) and view \( Ad G(e) \) as a coadjoint orbit in \( g^* \). Let \( \omega \) be the canonical symplectic 2-form on that orbit.

• Show that the 2-form \( \pi^* \omega \) on \( \overline{Ad G(e)} \) extends to a regular, possibly degenerate, 2-form \( \omega_X \) on \( X \).

• Show that in the case where \( g_i = 0 \) for all odd \( i \) the 2-form \( \omega_X \) is in fact nondegenerate.