Preliminary draft

ON THE GEOMETRY OF SYMPLECTIC RESOLUTIONS

VICTOR GINZBURG

CONTENTS

1. POISSON SCHEMES

1.1. **Basic definitions.** In this section, we work over an arbitrary field $\Bbbk$ of characteristic zero.

Definition. Let A be a commutative $\Bbbk$-algebra.

- A Poisson structure on A is the structure of a Lie algebra over $\Bbbk$ such that the Lie bracket $\{-,-\} : A \times A \to A$ satisfies the Leibniz identity:
$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b, \qquad \forall a, b, c \in A. \tag{1.1.1}$$
In such a case, one says that A is a Poisson algebra and $\{-,-\}$ is the corresponding Poisson bracket.
- The Poisson center of the Poisson algebra A is the center of A, viewed as a Lie algebra, i.e. the set of all $z \in A$ such that $\{z, a\} = 0$ for all $a \in A$.
- A Poisson ideal is an ideal I in A, viewed as a commutative algebra, such that one has $\{I, A\} \subset I$.

Equation (1.1.1) says that the map $\{-, c\} : A \to A$ is a derivation of A as a commutative algebra. By skew symmetry, the map $\{c, -\}$ is also a derivation of A, to be denoted ξ_c.

The Poisson center of A is a subalgebra in A.

If I is a Poisson ideal in A then the Poisson bracket on A descends to A/I and makes A/I a Poisson algebra such that the map $A \to A/I$ respects the brackets.

Given a commutative algebra A, write $\sqrt{A}$ and $\operatorname{Sing}(A)$, for the radical of A and the ideal of the singular locus of the scheme $\operatorname{Spec} A$, respectively.

Theorem 1.1.2. *Let A be a Poisson algebra and I a Poisson ideal in A.*

(i) *Any minimal prime ideal that contains a given Poisson ideal I is itself a Poisson ideal;*

(ii) *The ideals $\sqrt{A}$ and $\mathrm{Sing}(A)$ are Poisson ideals.*

(iii) *For any multiplicative subset $S \subset A \smallsetminus \{0\}$, the Poisson bracket on A has a unique extension to the localization $S^{-1}A$*

(iv) *If A is a domain then the Poisson bracket on A has a canonical extension to a Poisson bracket on the integral closure of A in its field of fractions.*

1.2. **Proof of Theorem 1.1.2.** We begin with some general results in commutative algebra. We let $Q(A)$ denote the field of fraction of a domain A.

Let A be an arbitrary finitely-generated $\Bbbk$-algebra and $\delta : A \to A$ a $\Bbbk$-derivation.

Lemma 1.2.1. *Let $I \subset A$ be an ideal. Define*

$$J = \left\{ x \in I \mid \delta^k(x) \in A, \ \ \forall\, k \geq 0 \right\}.$$

Clearly J is a δ-stable ideal contained in I. If I is prime, then J is prime.

Proof. Define a map

$$\exp:\ A \to A[[t]], \quad a \mapsto \sum_{i=0}^{\infty} \frac{t^i}{i!}\delta^i(a).$$

Recall that since δ is a derivation, $\exp$ is a ring homomorphism. Then $I[[t]] \subset A[[t]]$ is an ideal and we see $J = \exp^{-1}(I[[t]])$. Now I prime implies $A[[t]]/I[[t]] \cong A/I[[t]]$ has no zero divisors. So $I[[t]]$ is prime, so J is prime. □

Corollary 1.2.2. (i) *Let $\mathfrak{a} \subset A$ be a δ-stable ideal. Any minimal prime $I \supset \mathfrak{a}$ and hence $\sqrt{\mathfrak{a}}$ is also δ-stable.*

(ii) *Assume A is a domain and let $\overline{A}$ be the integral closure of A in $Q(A)$. Then δ extends uniquely to $Q(A)$ and $\overline{A}$ is δ-stable.*

Proof. (i) Let $I \subset \mathfrak{a}$ be a minimal prime and let J be as in the lemma. Since I is prime, J is prime, and since $\mathfrak{a}$ is δ-stable, $\mathfrak{a} \subset J$. So we have $\mathfrak{a} \subset J \subset I$ and because I is minimal $I = J$. Therefore I is δ-stable.

(ii) Recall that if B is integrally closed, then so is $B[[t]]$. It follows that $\overline{A[[t]]} = \overline{A}[[t]]$. Now we use the idea of the proof of Lemma 1.2.1. Since the map $\sum_i a_i t^i \mapsto a_0$ is left-inverse to $\exp$, we see The map $\exp$ is injective. Furthermore, the imbedding $\exp : A \hookrightarrow A[[t]]$. extends to $\exp : Q(A) \hookrightarrow Q(A[[t]])$. So $\exp(\overline{A}) \subset \overline{A[[t]]} = \overline{A}[[t]]$. So if $a \in \overline{A}$, then each coefficient of $\exp(a)$ is in $\overline{A}$. So $\delta^k(a) \in \overline{A}$ for all k. In particular $\delta(a) \in \overline{A}$ as desired. □

The following result seems to be standard. We give a proof[1] for the reader's convenience.

Lemma 1.2.3. *If O is a local $\Bbbk$-domain of Krull dimension 1 and R is its integral closure then for each local ring A of R, there is an element x in A such that $A = \mathsf{O}[x]$.*

Proof. Say $\mathfrak{m}$ is the maximal ideal of O. Suppose that $z \in A$ is a unit which descends to a primitive generator of the residue field of A over $\mathsf{O}/\mathfrak{m}$ and is such that $A = R[z]$. Let y be any uniformizer of A. We claim that $A = \mathsf{O}[z, y]$. First note that $\mathfrak{m}\mathsf{O}[z]$ is in the Jacobson radical of $\mathsf{O}[z]$. Indeed the inclusion of $\mathsf{O} \hookrightarrow \mathsf{O}[z]$ induces an isomorphism on fraction fields. So if a prime ideal $p \subset \mathsf{O}[z]$ satisfied $\mathsf{O} \cap p = 0$ then we have inclusions $Q(\mathsf{O}) \to \mathsf{O}[z]_p \to Q(\mathsf{O})$. Thus $p = 0$. We view the inclusion $\mathsf{O}[z, y] \hookrightarrow A$ as an embedding of $\mathsf{O}[z]$-modules. To check that it is surjective it suffices to check modulo $\mathfrak{m}\mathsf{O}[z]$ since $A = R[z]$ is finite over $\mathsf{O}[z]$ and $\mathfrak{m}\mathsf{O}[z]$ is in the Jacobson radical. So

[1]We are gratefull to Ian Shipman for providing this proof.

we have $\mathsf{O}[z,y]/\mathfrak{m}\mathsf{O}[z,y] \to A/\mathfrak{m}A$, which we view as a map of $\mathsf{O}/\mathfrak{m}[t]$-modules where t acts as y on both sides. Since t acts nilpotently on $A/\mathfrak{m}A$, it is supported at $t = 0$. So it suffices to check surjectivity by checking it modulo y. Now A/yA is a field and z projects to a primitive generator of it over $\mathsf{O}/\mathfrak{m}$. Hence the map is surjective modulo y as desired.

To see that z as above exist we note that $R/\mathfrak{m}R$ is an Artin ring. So it is the product of its localizations, one of which corresponds to A. We can find an element $a \in R$ such that

(1) the image of a is zero in all of the factors of $R/\mathfrak{m}R$ except the one corresponding to A
(2) a projects to a primitive generator of the residue field of A over O/m. Then $z = 1/a$ has the necessary properties.

Let P be a lift to $\mathsf{O}[t]$ of the minimal polynomial of the residue of z over $\mathsf{O}/\mathfrak{m}$. First, if $P(z)$ is a uniformizer then since $A = \mathsf{O}[z, P(z)] = \mathsf{O}[z]$ we can take $x = z$ as our generator. If $P(z)$ is not a uniformizer it must lie in $\mathfrak{m}_A^2$. We choose a uniformizer y and set $x = z + y$. This generates A over O since $P(x) = P(z + y) = P'(z)y \bmod (y^2)$ and P was minimal. □

Let A be a Poisson algebra and M be a module over A, viewed as a commutative algebra.

Definition. A Poisson A-module is an A-module M, where A is regarded as a commutative algebra, equipped with a Lie action $\{-,-\}_M : A \times M \to M, \quad a, m \to \{a, m\}_M$, satisfying properties similar to those for a Poisson algebra:

(1) $\{ab, m\}_M = a\{b, m\}_M + b\{a, m\}_M$,
(2) $\{a, bm\}_M = a, b \cdot m + a \cdot \{b, m\}_M$
(3) $\{\{a, b\}, m\}_M = \{a, \{b, m\}\}_M - \{b, \{a, m\}\}_M$.

Note that equation (2) says that for any $a \in A$ the map $\{a, -\}_M$ is a derivation of M as a module over a commutative algebra.

Note that any Poisson ideals of a Poisson are nothing but the Poisson submodules of that algebra, viewed as a module over itself. Using this, Theorem 1.1.2 can now be deduced from the following two results

Lemma 1.2.4. *Let $\delta : A \to A$ be a derivation of a commutative algebra A, let M be an A-module, and $\delta_M : M \to M$ a $\Bbbk$-linear map such that $\delta_M(am) = \delta(a)m + a\delta_M(m)$ for all $a \in A, m \in M$. Then the annihilator of M in A is a δ-stable ideal of A.*

Corollary 1.2.5. *Any derivation of A preserves the ideal* $\operatorname{Sing} A$.

The proof of part (iv) of the Theorem is reduced to extending Poisson structures in codimension 1 in certain cases. In those cases the argument is based on Lemma 1.2.3.

1.3. **Poisson geometry.** Let X be a scheme with structure sheaf $\mathcal{O}_X$, and $A = \Gamma(X, \mathcal{O}_X)$. We write

- $\Omega^1_X =$ sheaf of Kähler differentials; $\Omega^n_X := \wedge^n_{\mathcal{O}_X}\Omega^1_X$;
- $\mathcal{T}_X$ = sheaf of derivations $\mathcal{O}_X \to \mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ = 'vector fields'.
- $Poly^n_X$ = sheaf of $\Bbbk$-linear maps $\wedge^n\mathcal{O}_X \to \mathcal{O}_X$ which are derivations in each of of the arguments = $\mathcal{H}om_{\mathcal{O}_X}(\Omega^n\mathcal{O}_X, \mathcal{O}_X)$ = 'polyvector fields'.

We write $A = \Gamma(X, \mathcal{O}_X)$, resp. $\mathcal{T}(X) = \Gamma(X, \mathcal{T}(X))$, etc.

With this dictionary, giving a skew-symmetric pairing $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ that satisfies (1.1) is equivalent to giving a bivector field $\Pi \in Poly^2(X) = \operatorname{Hom}_{\mathcal{O}_X}(\Omega^2_X, \mathcal{O}_X)$ such that

$$\{a, b\} = \Pi(da \wedge db). \tag{1.3.1}$$

The above formula shows that the notion of Poisson structure is local, in particular, the RHS of (1.3.1) makes sense for local sections $a, b \in \mathcal{O}_X$ of the structure sheaf of an arbitrary scheme X.

Thus, there is a well-defined notion of Poisson bracket on $\mathcal{O}_X$. A scheme equipped with such a bracket is called a Poisson scheme.

The following result is an immediate consequence of Theorem 1.1.2

Proposition 1.3.2. *The reduction, normalization, and irreducible components of a Poisson scheme are themselves Poisson schemes.*

1.4. **The smooth case.** Assume now that X is smooth. Then $Poly^n_X = \wedge^n \mathcal{T}_X$. Here and below we often abuse the notation and write $\wedge^n \mathcal{T}_X$ for $\wedge^n_{\mathcal{O}_X} \mathcal{T}_X$, etc.

The graded $\mathcal{O}_X$-module $\mathrm{Sym}^\bullet \mathcal{T}_X$, resp. $Poly^n_X$, has the canonical structure of a Poisson algebra, resp. Poisson *super*-algebra, such that:

- multiplication:= commutative product on $\mathrm{Sym}^\bullet \mathcal{T}_X$, resp. $\wedge$-product on $\wedge^\bullet \mathcal{T}_X$.
- bracket is uniquely determined, via the Leibniz, resp. super-Leibniz, rule, by the formulas (same in both cases):

$$\{\xi, a\} := \xi(a), \qquad \{\xi, \eta\} := [\xi, \eta], \qquad \forall a \in \mathcal{O}_X, \xi, \eta \in \mathcal{T}_X.$$

The bracket $\{-,-\}$ on $Poly^n_X$ is called the Schouten bracket and will be denoted by $[-,-]_{\text{Scho}} : Poly^m_X \times Poly^n_X \to Poly^{m+n-1}_X$.

For $\Pi \in Poly^2(X)$ define the Lie derivative operator $L_\Pi = [\Pi, -]_{\text{Scho}} : Poly^\bullet_X \to Poly^{\bullet+1}_X$. This is a (super)-derivation wrt $\wedge$-product. One has

$$\text{Jacobi holds for (1.3.1)} \iff L_\Pi \text{ is a Lie derivation} \iff [\Pi, \Pi]_{\text{Scho}} = 0 \iff (L_\Pi)^2 = 0.$$

Corollary 1.4.1. *The bracket* (1.3.1) *is a Poisson bracket iff* $[\Pi, \Pi]_{Scho} = 0$. ☐

Example 1.4.2 (Constant brackets). Let $X = V$ be a vector space. Then any element $\Pi \in \wedge^2 V$ may be viewed as a constant bivector field on V. Explicitly, choose coordinates $x_1, \ldots, x_n$ on V. Then, one has

$$\Pi = \sum_{i,j,k} c^k_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Any such bivector satisfies Jacobi identity. The corresponding Poisson bracket on $\mathbb{C}[V]$ reads

$$\{f, g\} = \sum_{i,j} c_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Example 1.4.3 (Linear brackets). A linear bivector field on the vector space V is a bivector field of the form (we use the same notation as above):

$$\Pi = \sum_{i,j,k} c^k_{i,j} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad \text{so} \quad \{f, g\} = \langle \Pi, df \wedge dg \rangle = \sum_{i,j,k} c^k_{i,j} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

This bracket clearly preserves the space of linear functions, ie., we it has a property that $\{V^*, V^*\} \subset V^*$. Therefore, the above bracket is a Poisson bracket iff it gives V^* the structure of a Lie algebra with structure constants $c^k_{i,j}$. Conversely, any Lie algebra structure on a vector space $\mathfrak{g}$ with structure constants $c^k_{i,j}$ extends uniquely to a Poisson bracket on the algebra $\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\, \mathfrak{g}$, where $\mathfrak{g}$ stands for V^* in the previous notation.

Poisson ideals in $\mathrm{Sym}\, \mathfrak{g}$ are precisely the $\mathrm{ad}\, \mathfrak{g}$-stable ideals.

Associated with any $\Pi \in Poly^2(X)$, one has a contraction map

$$i_\Pi : \Omega^1_X \to \mathcal{T}_X, \tag{1.4.4}$$

This is a morphism of A-modules. Let $a \mapsto \xi_a$ be the composite map

$$i_\Pi \circ d: \ \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{i_\Pi} \mathcal{T}_X. \tag{1.4.5}$$

Then, the derivation of the commutative algebra $A = \mathbb{C}[X]$ that corresponds to the vector field ξ_a is the derivation $\{a, -\}$, by definition. In particular, we obtain

Corollary 1.4.6. *The bracket $\{-,-\}$ associated to a bivector field Π is a Poisson bracket iff the map $i_\Pi \circ d$ in* (1.4.5) *respects the brackets.*

The A-module morphism (1.4.5) can be extended uniquely to a graded A-algebra homomorphism

$$\sigma_\Pi : \Omega^\bullet_X = \wedge^\bullet_{\mathcal{O}_X} \Omega^1_X \to Poly^n_X = \wedge^\bullet_{\mathcal{O}_X} \mathcal{T}_X. \tag{1.4.7}$$

Proposition 1.4.8. (i) *The Jacobi holds for Π iff the map σ_Π intertwines the de Rham differential on $\Omega^\bullet_X$ with the map L_Π on $Poly^\bullet_X$.*

(ii) *If Π is a Poisson bivector then one has the following Tian-Todorov identity:*

$$[i_\Pi(\alpha), i_\Pi(\alpha)] = i_\Pi(d(i_\Pi(\alpha \wedge \beta))) - i_\Pi(d\alpha \wedge \beta) - i_\Pi(\alpha \wedge d\beta), \qquad \forall \alpha, \beta \in \Omega^1_X. \tag{1.4.9}$$

Hint for (i). Since both d and L_Π are super-derivations, it suffices to verify the statement for a set of generators of the algebra $\Omega^\bullet_X$. Thus, suffices to show that

$$\sigma_\Pi(d\alpha) = L_\Pi(\sigma_\Pi(\alpha))$$

holds in the cases where $\alpha = a \in A$ and $\alpha = db, \ b \in A$. □

Remark 1.4.10. The above constructions can be extended to the non-smooth case as well. In that case one has to replace the wedge-product by a $\cup$-product, resp. the Schouten bracket by the Gerstenhaber bracket. An analodue of the above proposition still holds but the proof is more difficult.

Definition. A symplectic form on a smooth variety X is a nondegenerate closed 2-form ω. In this case, one says that (X, ω) is a symplectic (algebraic) manifold.

A bivector Π is said to be nondegenerate if the map (1.4.4) is an isomorphism. In that case the 2-form $i_\Pi^{-1}(\Pi) \in \Omega^2_X$ is also nondegenerate. Thus, Proposition 1.4.8 yields

Corollary 1.4.11. *A nondegenerate bivector Π satisfies Jacobi iff the 2-form $\omega := i_\Pi^{-1}(\Pi) \in \Omega^2_X$ is a symplectic form.* □

Example 1.4.12. A calculation in (étale) local coordinates shows that the canonical Poisson structure on $\operatorname{Sym} \mathcal{T}_X$ is nondegenerate. So, for any smooth variety X, the cotangent bundle T^*X is a symplectic (algebraic) manifold.

1.5. **Symplectic leaves.** Given a bivector Π on a smooth variety X, the map (1.4.4) may be thought of as a morphism $\mathcal{T}^*_X \to \mathcal{T}_X$, from the cotangent to the tangent bundle on X. Let $V_x := \operatorname{Im}[T^*_x \to T_x]$ be the image of the corresponding linear map $i_{\Pi|_x} : T^*_x \to T_x$, of the fibers of $\mathcal{T}^*_X$ and $\mathcal{T}^*_X$ at $x \in X$. It follows from definitions that $\Pi|_x \in \wedge^2 V_x \subset \wedge^2 T_x$ and, moreover, $\Pi|_x$ viewed as an element of $\wedge^2 V_x$ is nondegenerate. Thus, the inverse of that element makes V_x a symplectic vector space.

The collection of spaces $V_x, \ x \in X$ is usually referred to as a distribution on X. It follows from Proposition 1.4.4 that the distribution is integrable, i.e. for any vector fields ξ, η tangent to the distribution, the vector field $[\xi, \eta]$ is tangent to the distribution as well. However, the dimension of the vector space V_x may depend on the point x, so our distribution does not necessarily have constant rank.

Assume that $\Bbbk = \mathbb{C}$. Then, according to a theorem of Frobebenius, an integrable holomorphic distribution on a complex manifold X gives a holomorphic foliation on X. Each leaf C is a smooth complex manifold such that for any $x \in C$ we have $T_xC = V_x$. Furthermore, the symplectic forms on the spaces V_x make C a holomorphic symplectic manifold. Note however that the natural imbedding $C \hookrightarrow X$ is not necessarily a closed imbedding: there may be leaves which are everywhere dense in X.

Now let X be a (possibly singular) complex algebraic variety and Π an algebraic Poisson bivector. We can partition X (set theoretically) into a union of smooth locally closed algebraic varieties inductively: $X_0 = X_{\text{reg}},\ X_1 := ((\text{Sing}(X))_{\text{red}})_{\text{reg}}, \ldots$. Each variety has a Poisson structure by Theorem 1.1.2, so we can consider symplectic leaves on X_i, viewed a smooth complex Poisson manifold. The collection of leaves of all the X_i's are the symplectic leaves of X, by definition.

Theorem 1.5.1. *If an algebraic Poisson variety has finitely many symplectic leaves then every such leaf is a locally closed algebraic subvariety.*

2. Hamiltonian reduction in the symplectic case

2.1. **Hamiltonian reduction of Poisson algebras. Hamiltonian reduction.** Let A be a Poisson algebra, $\mathfrak{g}$ a Lie algebra, and $\mathfrak{g} \to A$ a linear map that respects the Lie brackets. The canonical Poisson algebra structure on $\operatorname{Sym}\mathfrak{g}$ constructed in Example 1.4.3 has the following universal property:

Given a Poisson algebra A, any linear map $\rho : \mathfrak{g} \to A$ that respects the Lie brackets can be uniquely extended to a Poisson algebra homomorphism $\mu_\rho : \operatorname{Sym}\mathfrak{g} \to A$.

A Lie algebra map $\rho : \mathfrak{g} \to A$ gives a $\mathfrak{g}$-action on A by $x : a \mapsto \{\rho(x), a\}$. Given an $\operatorname{ad}\mathfrak{g}$-stable ideal $I \in \operatorname{Sym}\mathfrak{g}$, we consider $\mu_\rho(I) \cdot A$. This is a $\mathfrak{g}$-stable ideal of A in the sense of commutative algebras but it is not necessarily a Poisson ideal in A.

Claim 2.1.1. The Poisson bracket on A descends to a a well-defined operation on $(A/\mu_\rho(I) \cdot A)^{\mathfrak{g}}$. The resulting Poisson algebra is called the Hamiltonian reduction of A at I.

More geometrically, assume that $A = \mathbb{C}[X]$ and identify $\operatorname{Sym}\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$. Then the algebra homomorphism μ_ρ induces, and is induced by, a morphism $\mu : X \to \mathfrak{g}^*$, called the moment map associated to ρ. $\mathfrak{g}$-stable ideals $I \in \mathbb{C}[\mathfrak{g}^*]$ correspond to $\mathfrak{g}$-stable closed subschemes $Z \subset \mathfrak{g}^*$. Thus, we have $A/\mu_\rho(I) \cdot A = \mathbb{C}[\mu^{-1}(Z)]$. So, the Hamiltonian reduction algebra is $\mathbb{C}[\mu^{-1}(Z)]^{\mathfrak{g}}$, the subalgebra of $\mathfrak{g}$-invariant regular functions on $\mu^{-1}(Z)$.

Let G be a connected algebraic group G with Lie algebra $\mathfrak{g}$. In such a case, $\mathfrak{g}$-invariants=G-invariants. Thus, we have

$$(A/\mu_\rho(I) \cdot A)^{\mathfrak{g}} = \mathbb{C}[\mu^{-1}(Z)]^G = \mathbb{C}[\mu^{-1}(Z)//G], \tag{2.1.2}$$

where // denotes a categorical quotient by G.

Usually, one takes Z to be a closed coadjoint orbit, for example, a fixed point of the G-action on $\mathfrak{g}^*$. Observe that the differential of any Lie group homomorphism $G \to \mathbb{C}^\times$ gives a linear function $\lambda : \mathfrak{g} \to \mathbb{C}$ which is fixed by the coadjoint G-action. Conversely, if the group G is connected, then any Lie *algebra* homomorphism $\lambda : \mathfrak{g} \to \mathbb{C}$ is automatically fixed by the coadjoint G-action.

2.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let Y be a smooth manifold. Given a (smooth) G-action on Y, there is an associated Lie algebra map $\mathsf{act} : \mathfrak{g} \to$ *Vector fields on* Y. For $u \in \mathfrak{g}$, the value of the vector field $\mathsf{act}(u)$ at a point $y \in Y$ is the tangent vector $\mathsf{act}_y(u) \in T_yY$ corresponding to the 'infinitesimal u-action' on Y. Associated with the G-action on Y, there is a natural G-action on T^*Y.

The total space T^*Y, of the cotangent bundle has the canonical symplectic 2-form ω and the G-action on T^*Y respects the symplectic 2-form. Moreover, it is a Hamiltonian with moment map

$$\mu: T^*Y \to \mathfrak{g}^*, \quad \alpha \mapsto \mu(\alpha), \quad \text{defined by the equation } \langle \mu(\alpha), u\rangle = \langle \alpha, \mathsf{act}_y(u)\rangle, \tag{2.2.1}$$

for any $u \in \mathfrak{g}$, $y \in Y$, and any covector $\alpha \in T^*_y Y$. We write $d_\alpha\mu: T_\alpha(T^*Y) \to \mathfrak{g}^*$ for the differential of the moment map at the point α and $\mathsf{act}_\alpha^\top$ for the transpose of the linear map $\mathsf{act}_\alpha^\top: \mathfrak{g} \to T_\alpha(T^*Y)$.

The following properties of the map (2.2.1) are straightforward consequences of the definitions.

Proposition 2.2.2. (i) *The moment map is G-equivariant, i.e. it intertwines the G-action on T^*Y and the coadjoint G-action on $\mathfrak{g}^*$.*

(ii) *For any $\alpha \in T^*Y$, the following diagram commutes*

$$\begin{tikzcd} T_\alpha(T^*Y) \arrow[rr, "\omega", "\cong"'] \arrow[dr, "d_\alpha\mu"'] & & T^*_\alpha(T^*Y) \arrow[dl, "\mathsf{act}_\alpha^\top"] \\ & \mathfrak{g}^* & \end{tikzcd}$$

(iii) *Writing $T^*_Z Y$ for the conormal bundle of a submanifold $Z \subset Y$, one has*

$$\mu^{-1}(0) = \bigcup_{Z \in Y/G} T^*_Z Y. \tag{2.2.3}$$

Here, the horisontal map in the diagram of part (ii) is the isomorphism induced by the sympectic form and, in part (iii), Y/G stands for the set of G-orbits on Y.

Remark 2.2.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Every coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ comes equipped with a canonical G-equivariant symplectic 2-form ω (Kirillov-Kostant 2-form). The moment map for the G-action on $\mathcal{O}$ reduces to the tautological imbedding $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

An old result of Kostant says that, for any connected group G and a symplectic manifold X with a Hamiltonian transitive G-action, the corresponding moment map $\mu: X_0 \to \mathcal{O}$ must be a finite covering. $\lozenge$

From formula (2.2.3) one easily derives the following result.

Corollary 2.2.5. *Assume that the Lie group G acts freely on Y, and that the orbit space Y/G is a well defined smooth manifold. Then,*

- *The G-action on T^*Y is free, and the moment map* (2.2.1) *is a submersion.*
- *For any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, the orbit space $\mu^{-1}(\mathcal{O})/G$ has a natural structure of smooth symplectic manifold.*
- *For $\mathcal{O} = \{0\}$, there is, in addition, a canonical symplectomorphism*

$$T^*(Y/G) \cong \mu^{-1}(0)/G. \tag{2.2.6}$$

Formula (2.2.6) explains the importance of the zero fiber of the moment map. Later on, we will consider quotients of $\mu^{-1}(0)$ by a group G in situations where the group action on Y is no longer free, so the naive orbit set Y/G can not be equipped with a reasonable structure of a manifold. In those cases, various GIT type quotients of $\mu^{-1}(0)$ by G serve as substitutes for the contangent bundle on a nonexisting space Y/G.

More generally, let $\lambda \in \mathfrak{g}^*$ be a fixed point of the coadjoint action of G on $\mathfrak{g}^*$. Then, fiber $\mu^{-1}(\lambda)$ is necessarily a G-stable subvariety, by Proposition 2.2.2(i). The varieties of that form play the role of 'twisted cotangent bundles' on Y/G. These varieties share many features of the zero fiber of the moment map.

The above discussion was in the framework of differential geometry, where 'manifold' means a C^∞-manifold. There are similar constructions and results in the algebraic geometric framework where G now stands for a linear algebraic group and Y stands for a G-variety.

Example 2.2.7. Let G be a complex connected semisimple group with Lie algebra $\mathfrak{g}$ and let $\mathcal{O}$ be a nilpotent G-orbit in $\mathfrak{g}^* \cong \mathfrak{g}$. Further, let eu be the *Euler field*, the vector field on $\mathfrak{g}$ induced by the infinitesimal $\mathbb{C}^\times$-action on $\mathfrak{g}$ given by $\mathbb{C}^\times \ni z : x \mapsto z^2 \cdot x$. The Euler field is tangent to the variety $\mathcal{O}$ since any nilpotent G-orbit is $\mathbb{C}^\times$-stable, thanks to the Jacobson-Morozov theorem, cf. Example 5.2.1. Thus, $\beta := i_{\mathsf{eu}}\omega$ is a well-defined G-invariant 1-form on $\mathcal{O}$. Then, one easily checks that $\omega = d\beta$, so the symplectic form on any nilpotent orbit is an exact form. $\Diamond$

The following elementary result will be quite useful in applications to quiver varieties.

Lemma 2.2.8. *Let a connected group G act on a symplectic manifold M, and let μ be an associated moment map. Then, we have*

(i) *For $\alpha \in Y$, the following maps give a* selfdual *complex of vector spaces:*

$$\mathfrak{g} \xrightarrow{\mathsf{act}_\alpha} T_\alpha M \xrightarrow{d_\alpha\mu} \mathfrak{g}^*, \tag{2.2.9}$$

The symplectic form ω descends to a non-degenerate bilinear form on $\mathrm{Ker}(d_\alpha\mu)/\mathrm{Im}(\mathsf{act}_\alpha)$, the middle cohomology.

(ii) *Let $\lambda \in \mathfrak{g}^*$ be a fixed point of the coadjoint action of G and $\mu^{-1}(\lambda)$ the corresponding* scheme theoretic *fiber of μ. Then, $\alpha \in \mu^{-1}(\lambda)$ is a smooth point of the scheme $\mu^{-1}(\lambda)$ if and only if α has finite isotropy in G. In such a case, we have:*

- *In* (2.2.9), *the map act_α is injective and the map $d_\alpha\mu$ is surjective.*
- *The tangent space to the quotient variety $\mu^{-1}(\lambda)/G$ (if it exists) at the point corresponding to α is canonically isomorphic to $\mathrm{Ker}(d_\alpha\mu)/\mathrm{Im}(\mathsf{act}_\alpha)$ and we have:*

$$\dim(\mu^{-1}(\lambda)/G) \;=\; \dim M - 2\dim G.$$

Moreover, the symplectic form on M induces a symplectic 2-form on $\mu^{-1}(\lambda)/G$.

Proof. We note that the selfduality in part (i) refers to the isomorphism $T_\alpha M \cong T^*_\alpha M$ provided by the symplectic form. With this in mind, the selfduality statement is a direct consequence of Proposition 2.2.2(ii). Observe further that composite map $d_\alpha\mu \circ \mathsf{act}_\alpha$ may be identified with the infinitesimal coadjoint $\mathfrak{g}$-action on $\mathfrak{g}^*$. The point λ is fixed by the coadjoint action by the assumptions of the lemma. We conclude that $d_\alpha\mu \circ \mathsf{act}_\alpha = 0$ and part (i) follows.

To prove (ii), write $G^\alpha \subset G$ for the isotropy group of the point α. Clearly, we have $\mathrm{Lie}\, G^\alpha = \mathrm{Ker}(\mathsf{act}_\alpha)$. Thus, we deduce

$$\begin{aligned}
G^\alpha \text{ is finite} \quad &\Longleftrightarrow \quad \mathrm{Lie}\, G^\alpha = 0 \\
&\Longleftrightarrow \quad \mathrm{Ker}(\mathsf{act}_\alpha) = 0 \\
&\Longleftrightarrow \quad d_\alpha\mu \text{ is surjective} \quad \text{(by the selfduality in (i)).} \\
&\Longleftrightarrow \quad \alpha \text{ is a smooth point of } \mu^{-1}(\lambda).
\end{aligned}$$

Now, using that $d_\alpha\mu$, the differential of the moment map, is surjective by (i), we compute

$$\dim(\mu^{-1}(\lambda)/G) \;=\; \dim \mu^{-1}(\lambda) - \dim G \;=\; (\dim M - \dim G) - \dim G \;=\; \dim M - 2\dim G.$$

The last claim of part (ii) follows since the tangent space to $\mu^{-1}(\lambda)/G$ at the image of the point α is isomorphic to $T_\alpha(\mu^{-1}(\lambda))/T_\alpha(G\cdot\alpha)$. We leave details to the reader. $\square$

2.3. **Deustermaat-Heckman Theorem.**

Theorem 2.3.1. *The period map $\mathfrak{t}^* \to H^2(\overline{\mu}^{-1}(\chi))$ is an affine linear map, that is $d(\mathrm{Per})_\chi$ does not depend on the point χ.*

We consider the principal T-bundle $\overline{X}$ as in

$$X \xrightarrow{T} X/T = \overline{X} \xrightarrow{\overline{\mu}} \mathfrak{t}^*.$$

The first Chern class is an element $c_1(X, X/T) \in H^2(X/T) \otimes \mathfrak{t}$.

Claim 2.3.2. The map $d(Per)_\chi : \mathfrak{t}^* \to H^2(\mu^{-1}(\chi))$ is given by

$$d(Per)_\chi(\lambda) = \langle \lambda, c_1(X, X/T)|_{\mu^{-1}(\chi)} \rangle.$$

Note that since $X/T \to \mathfrak{t}^*$ is a smooth fiber bundle and $\mathfrak{t}^*$ is contractible the restriction maps $H^2(X/T) \to H^2(\mu^\chi)$ are isomorphisms for all $\chi \in \mathfrak{t}^*$. Here, the period map sends χ to $[\omega'|_{\mu^{-1}(\chi)}] \in H^2(\mu^{-1}(\chi)$, where ω' is the relative symplectic form $\omega' \in \Omega^2(\overline{X}/\mathfrak{t}^*)$.

Proof sketch for $T = \mathbb{C}^*$. Write $\bar{X} = X/T$. Since the action of T on X is free, $H_T^\bullet(X) = H^\bullet(\bar{X})$. We will use the de Rham construction of equivariant cohomology. Consider $(\Omega_X^\bullet[u])^T$ where u is a formal variable of degree r and $\Omega_X^\bullet$ is the sheaf of differential forms. We endow this with the differential $d_T = d_{dR} + ui_\xi$ where $\xi \in \Gamma(X, T_X)$ is an infinitesimal generator for the action. Now $H^\bullet(\Omega_X[u]^T, d_T) = H_T^\bullet(X)$. Of course $d_T(u) = 0$ and the image of $[u]$ in $H_T^\bullet(X) = H^\bullet(\bar{X})$ is the first Chern class of the bundle $X \to \bar{X}$, that is

$$\begin{aligned} H^\bullet(\Omega_X[u]^T, d_T) &\xrightarrow{\sim} H_T^\bullet(X) = H^\bullet(\bar{X}) \\ u &\longmapsto c_1[X \to \bar{X}]. \end{aligned}$$

Write $\frac{\partial}{\partial t}$ for the generator of $\mathfrak{t}^* = \mathbb{C}$ dual to ξ. Let $\omega \in \Omega^2(X)$ be the symplectic form and $\omega' \in \Omega^2(\bar{X}/\mathfrak{t}^*)$ be the relative symplectic form. Now we map

$$\begin{aligned} \omega' = Per : \mathfrak{t}^* &\longrightarrow H^2(\bar{\mu}^{-1}(\chi)) \\ \chi &\longmapsto [\omega'_\chi]. \end{aligned}$$

Then we identify $[\omega'_\chi]$ with a class in $H_T^2(\mu^{-1}(\chi))$. We may consider $\omega|_{\mu^{-1}(\chi)}$. Now

$$d_T\omega|_{\mu^{-1}(\chi)} = (d + ui_\xi)\omega|_{\mu^{-1}(\chi)} = ui_\xi\omega|_{\mu^{-1}(\chi)}.$$

Since the T-action is Hamiltonian, $i_\xi\omega = dH(\xi)$ and by definition $H(\xi) = \mu^*(t)$. Hence $i_\xi\omega = \mu^*(dt)$.

We recall the definition of the Gauss-Manin connection. To find $\frac{\partial\omega'}{\partial t}$ we lift ω' to a class ω in $\Omega_X^\bullet[u]^T$, differentiate it to $d_T\omega = u\mu^*dt$, then plug in $\mu^*\frac{\partial}{\partial t}$ giving u since $\langle \mu^*\frac{\partial}{\partial t}, \mu^*dt \rangle = 1$. This is the Chern class of the principal T-bundle $X \to \bar{X}$.

□

3. Deformations and quantizations of Poisson schemes

3.1. **Basic idea.**

The idea of *quantization* may be illustrated by the following example:

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Lie bracket $[-,-]$. be a Lie bracket on a finite dimensional vector space $\mathfrak{g}$. Then, for any scalar $\hbar \in \Bbbk$ the paring $[-,-]_\hbar := \hbar \cdot [-,-]$ is also a Lie bracket on the vector space $\mathfrak{g}$. So, we get a 1-parameter family $\mathfrak{g}_\hbar$ of Lie algebras $\mathfrak{g}_\hbar$ with the same underlying vector space and such that $\mathfrak{g}_{\hbar=1}$ is the original Lie algebra and $\mathfrak{g}_{\hbar=0}$ is an abelian Lie algebra. Therefore, the algebra $\mathcal{U}\mathfrak{g}$, the enveloping algebra of $\mathfrak{g}$, becomes a member of a 1-parameter family $\mathcal{U}\mathfrak{g}_\hbar$ of associative algebras such that $\mathcal{U}\mathfrak{g}_{\hbar=0} \cong \operatorname{Sym}\mathfrak{g}$ is a commutative algebra.

Intuitively, one views $\mathcal{U}\mathfrak{g}$ as a noncommutative deformation, aka quantization, of the commutative algebra $\operatorname{Sym}\mathfrak{g}$.

The notion of 'deformation' may be formalized as follows.

Definition. Fix an associative (not necessarily commutative) $\mathbb{k}$-algebra A_0. Let R be a finitel
y generated commutative $\mathbb{k}$-algebra and $\mathfrak{m}$ a $\mathbb{k}$-point of $\operatorname{Spec} R$, i.e. a maximal ideal of R such that $R/\mathfrak{m} = \mathbb{k}$.

- A flat deformation of A_0 over R is a flat associative R-algebra A equipped with an isomorphism $A/\mathfrak{m}A \cong A_0$, of $\mathbb{k}$-algebras.
- An n-th order 1-parameter deformation of A_0 is a flat (equivalently, free) $\mathbb{k}[t]/(t^{n+1})$-algebra A equipped with an isomorphism $A/\mathfrak{m}A \cong A_0$, of $\mathbb{k}$-algebras.
- A formal 1-parameter deformation of A_0 is an algebra of the form $\operatorname{lim\,proj}_{n\to\infty} A_n$, an inverse limit of $\mathbb{k}[t]$-algebras such that A_n is an n-th order 1-parameter deformation of A_0. Giving a formal 1-parameter deformation of A_0 amounts to giving a flat $\mathbb{k}[[t]]$-algebra A, which is complete in the (t)-adic topology and is equipped with an isomorphism $A/tA \cong A_0$, of $\mathbb{k}$-algebras.

A fundamental relation between 1-parameter deformations and Poisson algebras is provided by following result

Proposition 3.1.1. *Let A be a first order deformation of a* commutative *algebra A_0. Then, we have*

(i) *The assignment*

$$A \times A \to A, \quad a \times b \mapsto \left(\tfrac{ab-ba}{t}\right) \bmod tA.$$

descends to a well-defined map $A/tA \times A/tA \to A/tA$. Transporting this map via the given isomorphism $A/tA = A_0$ one obtains a skew-symmetric bilinear pairing $\{-,-\} : A_0 \times A_0 \to A_0$ that satisfies the Leibniz identity (1.1.1)*.*

(iii) *If the first order deformation A can be lifted to a second order deformation, then the Jacobi identity holds, so the pairing $\{-,-\}$ gives A_0 the structure of a Poisson algebra.*

To put the above example with enveloping algebras in the setting of Definition 3.1 we let $\hbar$ be an indeterminate, write $T\mathfrak{g}$ for the tensor algebra of the vector space $\mathfrak{g}$, and put

$$\mathcal{U}_\hbar\mathfrak{g} := \frac{T\mathfrak{g} \otimes \mathbb{k}[\hbar]}{(x \otimes y - y \otimes x - \hbar \cdot [x,y],\ x, y \in \mathfrak{g})}$$

The algebra $\mathcal{U}_\hbar\mathfrak{g}$ is a flat 1-parameter deformation of $\operatorname{Sym}\mathfrak{g}$ over $R := \mathbb{k}[\hbar]$, which is a rigorous substitute for the 'family' $\mathcal{U}\mathfrak{g}_\hbar$. In particular, the construction of Proposition 3.1.1 gives $\operatorname{Sym}\mathfrak{g}$ the canonical Poisson algebra structure. The corresponding Poisson bracket satisfies $\{x,y\} := [x,y]$ for any $x, y \in \mathfrak{g}$, and it is uniquely determined by this formula via the Leibniz rule.

The above setting of 1-parameter quantizations is quite restrictive. It is not very obvious how to generalize it in the most correct way. Think of the following two *different* reasonable settings for deformation problem in Number theory. Fix a scheme X over a finite field $\mathbb{F}_p$. Then, one can look for:

1) Flat families $f : \mathcal{X} \to \operatorname{Spec}\mathbb{F}_p[[t]]$. with an isomorphism $f^{-1}(0) \cong X$.

or

2) Lifts of X to a flat scheme over $\mathbb{Z}_p$, the p-adic integers.

These problems have very different solutions.

In the quantization setting, we would like to mimic the second of the above deformation problems as follows.

3.2. **Deformations in the complex analytic setting.** Throughout the paper, we write $H^\bullet(-) := H^\bullet(-,\mathbb{C})$ for cohomology with $\mathbb{C}$-coeffitients. We use similar notation $H_\bullet(-) = H_\bullet(-,\mathbb{C})$ for homology.

Let X be a C^∞-manifold. A holomorphic structure on X determines and is determined by a $\bar\partial$-operator satisfying an integrability condition $\bar\partial^2 = 0$. The corresponding sheaf of holomorphic functions is then defined as a subsheaf of the sheaf of C^∞-functions formed by the functions annihilated by $\bar\partial$.

Now, fix a holomorphic structure on X. Let $\mathcal{O}_{hol}$, resp. $\mathcal{T}_{hol}$ and $\mathcal{T}^*_{hol}$, be the sheaf of holomorphic functions, resp. holomorphic tangent and cotangent sheaf, on X. The space of differential C^∞-forms on X of degree n has the (p,q)-decomposition $\bigoplus_{p+q=n} \Omega^{p,q}(X)$. Given a coherent sheaf $\mathcal{F}$, of $\mathcal{O}_{hol}$-modules, we let $\Omega^{p,q}(\mathcal{F})$ denote the space (possibly infinite dimensional) of global sections of the sheaf $\mathcal{F}\otimes_{\mathcal{O}_{hol}} \Omega^{p,q}_{hol}$. The $\bar\partial$-operator corresponding to our holomorphic structure gives, for each p, the Dolbeault complex

$$\Omega^{p,\bullet}:\ \ 0\to\Omega^{p,0}(\mathcal{F})\xrightarrow{\bar\partial}\Omega^{p,1}(\mathcal{F})\xrightarrow{\bar\partial}\Omega^{p,2}(\mathcal{F})\to\dots.$$

The Dolbeault theorem says that

$$H^q(X,\mathcal{F})\cong H^q(\Omega^{0,\bullet}(\mathcal{F}),\bar\partial) \tag{3.2.1}$$

A formal 1-parameter deformation of the given holomorphic structure on X is determined by a deformed $\bar\partial$-operator of the form $\bar\partial_t = \bar\partial + \phi_t$, where $\phi_t = \sum_{n=1}^\infty\ t^n\phi_n$ is a formal power series with coefficients $\phi_n \in \Omega^{0,1}(\mathcal{T}_{hol})$. So, in a local holomorphic coordinates each of the ϕ_n's has the form

$$\sum_{i,j,k} a_{i,j,k}(z,\bar z)\frac{\partial}{\partial z_i}dz_j d\bar z_j$$

we have where $a_{i,j,k}$ are some C^∞-functions. Since $\bar\partial^2 = 0$, the integrability condition for the operator $\bar\partial_t$ reads

$$0 = \bar\partial_t^2 = \bar\partial\phi + \tfrac{1}{2}[\phi,\phi],\quad \phi\in\Omega^{0,1}(\mathcal{T}_{hol}). \tag{3.2.2}$$

Now, let $\Pi\in\wedge^2\mathcal{T}_{hol}$ be a holomorphic bivector. Contraction with Π gives a morphism

$$i_\Pi : H^1(X,\Omega^1_{hol})\to H^1(X,\mathcal{T}_X). \tag{3.2.3}$$

In terms of Dolbeault complexes contraction with Π gives maps

$$\sigma:\ \Omega^{p+1,q}(X) = \Omega^{p,q}(\mathcal{T}^*_{hol})\to\Omega^{p,q}(\mathcal{T}_{hol}) = \Omega^{p-1,q}(X), \forall p,q.$$

In particular, for $\nu\in\Omega^{1,1}(X)$ we have $i_\Pi\nu\Omega^{0,1}(\mathcal{T}_{hol})$.

Theorem 3.2.4. *Let X be a complex manifold such that $H^2(X,\mathcal{O}_{hol}) = 0$ and Π a holomorphic Poisson bivector on X. Then, for any class $[\phi_1]\in H^1(X,\mathcal{T}_X)$ in the image of the map* (3.2.3) *there exists a formal deformation $\bar\partial_t = \bar\partial + \sum_{n=1}^\infty\ t^n\phi_n$ of the complex structure such that the formula*

$$\{f,g\}_t := \langle\Pi, df\wedge dg\rangle$$

gives a Poisson bracket on $C^\infty(X)[[t]]$ which is holomorphic with respect to the complex structure $\bar\partial_t$ for 'all t'.

Furthermore, if $H^1(X,\mathcal{O}_{hol}) = 0$ then such a deformation is unique up to isomorphism.

Proof. The assumption that $H^2(X,\mathcal{O}_{hol}) = 0$ implies that any class in $H^1(X,\Omega^1_{hol})$ has a Dolbeault representative $\nu\in\Omega^{1,1}$ such that $d\nu = 0$. Thus, we may assume that the class $[\phi_1]$ in the theorem is represented by $\sigma(\nu)$ such that $d\nu = 0$, that is, $\partial\nu = \bar\partial\nu = 0$.

We first find ϕ_t that satisfies (3.2.2). Expanding each side of this equation in powers of t and taking the linear term in t we see that we must have that $\bar\partial\phi_1 = 0$. This is indeed true since

$\bar\partial\phi_1 = \bar\partial\sigma(\nu) = \sigma(\bar\partial\nu) = 0$, because $\bar\partial$ commutes with σ. Taking terms involving t^2 gives the equation for each

$$\bar\partial\phi_2 = [\phi_1, \phi_1] = [\sigma(\nu), \sigma(\nu)]. \tag{3.2.5}$$

To find ϕ_2, we apply (1.4.9) in the case $\alpha = \beta = \nu$. To be able to do so, we treat all $\bar z$-variables as auxiliary parameters, replace d by ∂, and view ν as a 1-form wrt z-variables. Since $\partial\nu = 0$, the 2d and 3d terms in the RHS of the identity in (1.4.9) vanish and we obtain

$$[\phi_1, \phi_1] = [\sigma(\nu), \sigma(\nu)] = \sigma(\partial(\sigma(\nu^2))).$$

Note that ν being a form of total degree 2, the 4-form $\nu^2 := \nu\wedge\nu$ is not necessarily zero. Note also $\sigma(\nu^2) \in \Omega^{0,2}(X)$, so $\sigma(\partial(\sigma(\nu^2))) \in \Omega^{1,2}(\mathcal{T}_{hol})$ which is the degree we want. Comparing with (3.2.5) we see that it remains to show that $\sigma(\partial(\sigma(\nu^2))$ is $\bar\partial$-exact, i.e. that there exists $\phi_2 \in \Omega^{1,1}(X)$ such that $\sigma(\nu^2) = \bar\partial\phi_2$. Since $\bar\partial$ commutes with σ and $\bar\partial\nu = 0$ we get $\bar\partial(\sigma(\nu^2)) = \sigma(\bar\partial(\nu^2)) = 2\sigma(\bar\partial\nu\wedge\nu) = 0$. Hence, using that $H^2(\mathcal{O}_{hol}) = 0$ and applying the Dolbeault theorem we deduce that any $\bar\partial$-closed $(0,2)$-form is $\bar\partial$-exact. Hence one can find β such that $\sigma(\nu^2) = \bar\partial\beta$. We put $\phi_2 = \sigma\partial\beta$. Thus, using that $\bar\partial$ commutes with both ∂ and σ, we find

$$\bar\partial\phi_2 = \bar\partial\sigma\partial\beta = \sigma\partial\bar\partial\beta = \sigma\partial\sigma(\nu^2) = [\phi_1, \phi_1],$$

as required.

The main trick in solving (3.2.2) for ϕ_t is to look for solutions where each $\phi_n, n \geq 2$ is of the form $\phi_n = \sigma(\partial\beta_n)$ for some $\beta_n \in \Omega^{0,1}(X)$. Note that ϕ_2 does have such a form with $\beta_2 = \beta$. So, write $\beta_t = \sum_{n\geq 0} t^n \cdot \beta_{n+2}$ and let $\phi_t = t\sigma\nu + t^2\sigma\partial\beta_t$. We may treat the $\bar z$-variables as auxiliary parameters, β_t as a degree zero form in the z-variables. Then, Corollary 1.4.6 with d replaced by ∂ applies to such a degree zero form and we deduce $[\sigma\partial\beta_t, \sigma\partial\beta_t] = \sigma\partial(\{\beta_t, \beta_t\})$. So, to solve (3.2.2) for ϕ_t it is sufficient to solve

$$\bar\partial\beta_t = \tfrac{1}{2}\{\beta_t, \beta_t\}. \tag{3.2.6}$$

We solve this by finding β_n recursively by induction on n, where the case $n = 2$, the base of induction, has been done already.

So, assume we have constructed all the $\beta_k,\ k = 2, ..., n-1$, and for any $m \leq n$ put

$$\gamma_m := \{\beta_1, \beta_{m-1}\} + \{\beta_2, \beta_{m-2}\} + \ldots + \{\beta_{m-1}, \beta_1\}.$$

Note, that we have

$$\bar\partial\{\beta_i, \beta_j\} = \bar\partial\sigma(\partial\beta_i\wedge\partial\beta_j) = \sigma(\bar\partial\partial\beta_i\wedge\partial\beta_j) - \sigma(\partial\beta_i\wedge\bar\partial\partial\beta_j).$$

Using the definition of γ_m and induction, we know that for all $m < n$ we have $\bar\partial\partial\beta_m = -\frac{1}{2}\partial\gamma_k$. Inserting this in the definition of γ_n we find

$$\bar\partial\gamma_n = \sum_{i+j+k=n} \{\beta_i,\ \{\beta_j, \beta_k\}\}.$$

The RHS here vanishes by the Jacobi identity. We deduce that γ_n is $\bar\partial$-closed and hence is $\bar\partial$-exact since $H^2(X, \mathcal{O}_{hol}) = 0$. Thus, we can find β_n such that $\gamma_n = 2\bar\partial\beta_n$, that is, such that

$$\bar\partial\beta_n = \tfrac{1}{2}\big(\{\beta_1, \beta_{n-1}\} + \{\beta_2, \beta_{n-2}\} + \ldots + \{\beta_{n-1}, \beta_1\}\big).$$

This complete the induction step of solving (3.2.6). Thus, we have constructed a complex deformation $\bar\partial_t = \bar\partial + \phi_t$.

Next, we not that since Π is holomorphic, for any holomorphic functions f, g, one has $\langle\Pi, \partial f, \partial g\rangle = \langle\Pi, df\wedge dg\rangle$. The proof of the theorem is completed by showing that for any $f, g \in C^\infty(X)[[t]]$ such that $\partial_t f = 0$ and $\partial_t g = 0$ one also has $\partial_t(\langle\Pi, df\wedge dg\rangle) = 0$. $\square$

3.3. **Deformations of smooth symplectic algebraic varieties.** Let X be a smmoth algebraic symplectic variety with sympletic form $\omega \in H^0(X, \Omega_X^2)$.

Definition. A symplectic deformation of X over S, a local Artin scheme, is a deformation $\mathcal{X}$ of X over S equipped with a closed relative two form $\omega_{\mathcal{X}} \in H^0(\mathcal{X}, \Omega^2_{\mathcal{X}/S}$ and a symplectic isomorphism $(\mathcal{X}_0, \omega_0) \cong (\mathcal{X}, \omega)$.

Definition. Let *Def*(S) be the set of isomorphism classes of symplectic deformations of X over S. Using base change, $S \mapsto$ *Def*(S) becomes a functor from the category of local Artin schemes to sets.

Theorem 3.3.1. *If $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ then the functor $S \mapsto$ Def(S) is pro-representable by a formal scheme the* period map *Per that sends the symplectic form* Def. *Furthermore, sending the symplectic form to its de Rham cohomology class gives an isomorphism*

$$\mathrm{Def} \xrightarrow{\sim} \widehat{H^2(X, \mathbb{C})}_\omega,$$

where $\widehat{H^2(X, \mathbb{C})}_\omega$ is a formal completeion of the vector space $H^2(X, \mathbb{C})$ at the point $[\omega] \in H^2(X, \mathbb{C})$.

Remark 3.3.2. In the more general case where the groups $H^i(X, \mathcal{O}_X)$ do not vanish for $i = 1, 2$, the space $H^2(X, \mathbb{C})$ has to be replaced by another space involving Hodge filtrations. In more detail, let $F^\bullet \Omega_X^\bullet$ denote the Hodge filtration on the de Rham complex defined

$$(F^i \Omega_X^\bullet)^j = \begin{cases} \Omega_X^j & j \geq i, \\ 0 & j < i. \end{cases}$$

Observe that the form ω defines a morphism $\mathcal{O}_X[-2] \to F^1 \Omega_X^\bullet$ in the derived category of sheaves of k-vector spaces on X. Taking $\mathbb{H}^2$ of this map we get a map

$$H^0(X, \mathcal{O}_X) = \mathbb{H}^2(X, \mathcal{O}_X[-2]) \longrightarrow \mathbb{H}^2(X, F^1 \Omega_X^\bullet),$$
$$1 \longmapsto \omega$$

where we are denoting the image of 1 as ω by abuse of notation. Then the target of the period map should be taken to be a completion of $\mathbb{H}^2(X, F^1 \Omega_X^\bullet)$ at ω.

4. Symplectic singularities

4.1. **Nice $\mathbb{C}^\times$-actions.** The following class of algebraic varieties will play an important role in these lectures.

Definition. A *nice action* is a $\mathbb{C}^\times$-action $\mathbb{C}^* \times X \to X, (z, x) \mapsto zx$ on a quasi-projective variety X such that $X^{\mathbb{C}^\times}$, the fixed point set, is projective and the limit $\lim_{z \to 0} zx$ exists (i.e. the map $\mathbb{C}^\times \to X,\ z \mapsto zx$ can be extended to a map $\mathbb{P}^1 \to X$) for every $x \in X$, i.e. the map $\mathbb{C}^\times \to X,\ z \mapsto zx$.

- Giving an affine algebraic variety X with a $\mathbb{C}^\times$-action is equivalent to giving a $\mathbb{Z}$-graded finitely generated commutative algebra $A = \mathbb{C}[X]$. Then, the action is nice iff the grading $A = \oplus_{i \in \mathbb{Z}} A_i$ is nonnegative, i.e., $A_i = 0$ for all $i < 0$ and, moreover, we have $A_0 = \mathbb{C}$. In such a case, the point $o \in X$ that corresponds to the augmentation ideal $A_+ = \oplus_{i > 0} A_i$ is the unique $\mathbb{C}^\times$-fixed point in X, referred to as the 'origin', and $\lim_{z \to 0} zx = o$ for any $x \in X$.
- In general, for any, not necessarily affine, quasi-projective variety, let $\mathbb{C}[X] := \Gamma(X, \mathcal{O}_X)$. The affine variety $X^{\mathtt{aff}} := \operatorname{Spec} \mathbb{C}[X]$ is called the affinization of X. One has a canonical affinization morphism $X \to X^{\mathtt{aff}}$. A $\mathbb{C}^\times$-action on X induces a $\mathbb{Z}$-grading on $\mathbb{C}[X]$, hence, a $\mathbb{C}^\times$-action on $X^{\mathtt{aff}}$. If the action on X is nice then so is the action on $X^{\mathtt{aff}}$. Morever, the set $X^{\mathbb{C}^\times}$ maps to the point $o \in X^{\mathtt{aff}}$.

In the opposite direction, let X be a quasi-projective variety with a $\mathbb{C}^\times$-action such that the the action on $X^{\mathtt{aff}}$ is nice and the affinization morphism $X \to X^{\mathtt{aff}}$ is *proper*. Then, the action on X is nice.

- A special case of the above is the case where X is a $\mathbb{C}^\times$-equivariant resolution of singularities of an affine normal variety Y with a nice action. This case will be most important for us.

Recall that the fixed point set of a $\mathbb{C}^\times$-action on a smooth variety is a smooth closed subvariety. Thus, $F := \widetilde{X}^{\mathbb{C}^\times} \subset \widetilde{X}$ is a smooth closed subvariety. Write $F_1, \ldots, F_r$ for the connected components of F, and introduce the following sets

$$\Lambda_s := \{z \in \widetilde{X} \mid \lim_{t\to\infty} t(z) \text{ exists, and we have } \lim_{t\to\infty} t(z) \in F_s\}, \quad s = 1, \ldots, r. \tag{4.1.1}$$

Proposition 4.1.2. *Any smooth variety with a nice action has a Bialynicki-Birula decomposition $X = \sqcup \Lambda_s$ such that Λ_s are smooth locally closed subvarieties of X and the map $x \mapsto \lim_{z\to\infty} z(x)$ gives a morphism $\Lambda_s \to F_s$ which is isomorphic to a vector bundle on F_s.*

Proof. It was proved in [Bi] that, for a $\mathbb{C}^\times$-action on a smooth projective variety, each Bialynicki-Birula piece is a smooth, connected, and locally closed subvariety. Our variety $\widetilde{X}$ is not assumed to be projective, in general. Therefore, one first has to consider a compactification of $\widetilde{X}$. Specifically, it is known that one can always find a smooth projective variety Y, with a $\mathbb{C}^\times$-action, that contains $\widetilde{X}$ as a Zariski open and dense, $\mathbb{C}^\times$-stable subvariety. Now, we have seen that the $\mathbb{C}^\times$-action on $\widetilde{X}$ is a contraction to $F = \widetilde{X}^{\mathbb{C}^\times}$. It follows that each Bialynicki-Birula piece for the $\mathbb{C}^\times$-action on Y is either entirely contained in $\widetilde{X}$ or in $Y \smallsetminus \widetilde{X}$. Thus, $\widetilde{X}$ is a union of a certain subcollection of pieces of the Bialynicki-Birula decomposition of Y. ☐

4.2. Let Y be a (possibly singular) variety equipped with an algebraic Poisson structure. In algebraic terms, this means that $\mathbb{C}[Y]$, the coordinate ring of Y, is equipped with a Poisson bracket $\{-,-\}$, that is, with a Lie bracket satisfying the Leibniz identity.

Let Y be a normal variety, let Y^{reg} denote the *smooth locus* of Y, and let ω^{reg} be an algebraic symplectic 2-form on Y^{reg}. Since $\Gamma(Y^{\text{reg}}, \mathcal{O}_{Y^{\text{reg}}}) = \Gamma(Y, \mathcal{O}_Y) = \mathbb{C}[Y^{\mathtt{aff}}]$, the Poisson bracket on $\Gamma(Y^{\text{reg}}, \mathcal{O}_{Y^{\text{reg}}})$ induced by the symplectic structure gives the affinization $Y^{\mathtt{aff}}$ the structure of a Poisson variety. Following Beauville [Bea], one says that Y has *symplectic singularities* if there exists a resolution (equivalently, for any resolution) of singularities $\pi : \widetilde{Y} \to Y$ such that the 2-form $\pi^*(\omega^{\text{reg}})$, on $\pi^{-1}(Y^{\text{reg}})$, extends to a regular (possibly degenerate) 2-form on the whole of $\widetilde{Y}$.

It turns out that resolutions of a Poisson variety X with symplectic singularities enjoy a number of very favorable properties. Specifically, one has the following result.

Theorem 4.2.1. *Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of an affine variety X with symplectic singularities, and $\omega \in \Omega^2(\widetilde{X})$ the regular extension to $\widetilde{X}$ of the pull-back of the symplectic form on X^{reg}. Then, the following holds:*

(1) *The variety X has rational singularities, equivalently, we have $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for all $i > 0$.*

(2) *For any $x \in X$, the fiber $\widetilde{X}_x$ is an isotropic subvariety of $\widetilde{X}$, i.e., the restriction of ω to the smooth locus of any irreducible component of $\widetilde{X}_x^{red}$ vanishes.*

(3) *Any algebraic action on X of a connected and simply connected semisimple group G can be canonically lifted to a G-action on $\widetilde{X}$.*

(4) *The Poisson variety X is a union of* finitely many *symplectic leaves $X = \sqcup X_\alpha$, [K4], and each symplectic leaf X_α is a locally closed smooth algebraic subvariety of X, [BG].*

Proof. First of all, we observe that since X is normal we have $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$). Also, one has $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^0(X, R^i\pi_*\mathcal{O}_{\widetilde{X}})$, since X is affine. Part (1) follows from these observations thanks to a general theorem by Elkik [El]. For completeness, we provide a self-contained and streamlined proof of that result in Appendix 10.

To prove (2) one uses the following argument due to Wierzba [W] (extended and completed by Namikawa [N1]).

Let $\overline{\omega}$ be the complex conjugate of the 2-form ω. Thus, $\overline{\omega}$ is an anti-holomorphic 2-form that gives a Dolbeault cohomology class $[\overline{\omega}] \in H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. The latter class is in fact equal to zero since we have shown that $H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$.

Next let $x \in X$. We must prove that the restriction of the 2-form ω, equivalently, the restriction of the 2-form $\overline{\omega}$, to $\pi^{-1}(x)$ vanishes. To this end, let $Y \twoheadrightarrow \pi^{-1}(x)$ be a resolution of singularities of the fiber, and write $f : Y \to \widetilde{X}$ for the composite $Y \twoheadrightarrow \pi^{-1}(x) \hookrightarrow \widetilde{X}$. Thus, $f^*\overline{\omega}$ is an anti-holomorphic 2-form on Y and, in Dolbeault cohomology of Y, we have $[f^*\overline{\omega}] = f^*[\overline{\omega}] = 0$.

On the other hand, Y is a smooth and projective variety. Hence, by Hodge theory, we have $H^2(Y, \mathcal{O}_Y) \xrightarrow{\sim} H^{0,2}(Y) \subset H^2(Y)$. It is clear that the Dolbeault cohomology class of the 2-form $f^*\overline{\omega}$ goes, under this isomorphism, to the de Rham cohomology class of $f^*\overline{\omega}$. Thus, in the de Rham cohomology of Y, we have $[f^*\overline{\omega}] = 0$. But any nonzero anti-holomorphic differential form on a Kähler manifold is harmonic, hence gives a nonzero de Rham cohomology class, thanks to Hodge theory. It follows that the 2-form $f^*\overline{\omega}$ vanishes. We deduce that $\overline{\omega}|_{\pi^{-1}(x)} = 0$, and we are done.

Part (3) is a consequence of [GK, Lemma 5.3]. That Lemma implies that the G-action on X can be lifted to an infinitesimal action on $\widetilde{X}$ of the Lie algebra $\operatorname{Lie} G$. The assumptions on the group G made in the statement of Theorem 4.2.1(3) then insure that the Lie algebra action can be exponentiated to an action of the group G.

Part (4) is much harder. We refer to [K4] for a proof. □

4.3. Given an affine Poisson variety X with a nice $\mathbb{C}^\times$-action $z : x \mapsto z(x)$, on X, we say that it is nice wrt to the Poisson structure if there is a (fixed) positive integer $m > 0$ such that, under the action of any element $z \in \mathbb{C}^\times$, the Poisson bivector on X gets rescaled with weight z^{-m}.

Let X be a normal affine Poisson variety with symplectic singularities equipped with a nice $\mathbb{C}^\times$-action. Let $o \in X$ be the unique $\mathbb{C}^\times$-fixed point.

Let $\pi : \widetilde{X} \to X$ be a resolution of singularities. According to Theorem 4.2.1(3), the $\mathbb{C}^\times$-action on X has a canonical lift to a $\mathbb{C}^\times$-action on $\widetilde{X}$ that makes π a $\mathbb{C}^\times$-equivariant morphism. The fiber $\widetilde{X}_o = \pi^{-1}(o)$ is usually referred to as the *central fiber*. It is clear that $\widetilde{X}_o$ is a $\mathbb{C}^\times$-stable projective subvariety of $\widetilde{X}$, Furthermore, the $\mathbb{C}^\times$-action provides a homotopy retraction of $\widetilde{X}$ to $\widetilde{X}_o$. In particular, we have $H^\bullet(\widetilde{X}) \cong H^\bullet(\widetilde{X}_o)$.

4.4. Let X be an affine normal Poisson variety with a nice $\mathbb{C}^\times$-action. Observe that condition (**??**) insures that the Poisson bracket makes the vector space $\mathfrak{g}_X := \mathbb{C}^m[X]$, the nonzero homogeneous component of minimal positive degree, a finite dimensional Lie algebra. This Lie algebra acts on the commutative algebra $\mathbb{C}[X]$ by derivations $\{g, -\},\ g \in \mathbb{C}^m[X]$. It is clear that each homogeneous component $\mathbb{C}^i[X]$ is $\mathfrak{g}_X$-stable. It follows that the $\mathfrak{g}_X$-action on $\mathbb{C}^i[X]$ can be exponentiated to an action of a connected algebraic group G with Lie algebra $\mathfrak{g}_X$. Thus, one gets a G-action on $\mathbb{C}[X]$ by algebra automorphisms. The resulting G-action on the variety X is automatically Hamiltonian with moment map $\mu : X \to \mathfrak{g}_X^*$ being the evaluation map $\langle \mu(x), g\rangle = \jmath(g)(x)$ for all $x \in X$ and $g \in \mathfrak{g}_X$, where $\jmath :\ \mathfrak{g}_X \hookrightarrow \mathbb{C}[X]$ is the tautological imbedding.

Interesting examples of the above construction arise from quiver varieties, cf. M. Finkelberg and D. Kubrak [FK]. Another class of examples is provided by the following theorem which is, essentially, a combination of results of the papers [BK], [Fu] and [Pa]

Theorem 4.4.1. *The variety X contains an open dense G-orbit if and only if there exists a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ and a finite G-equivariant covering $\widetilde{\mathcal{O}} \twoheadrightarrow \mathcal{O}$, such that X is isomorphic to $\operatorname{Spec} \mathbb{C}[\widetilde{\mathcal{O}}]$ as a Poisson $G \times \mathbb{C}^\times$-variety, where $\mathbb{C}^\times$ acts on $\mathfrak{g}$ by $z : g \mapsto z^m \cdot g$. In such a case, the following holds:*

(i) *The Lie algebra $\mathfrak{g}_X = \mathbb{C}^m[X]$ is semisimple.*

(ii) *The moment map μ is a* finite *morphism with image $\overline{\mathcal{O}}$, the Zariski closure of the G-orbit $\mathcal{O} \subset \mathfrak{g} = \mathfrak{g}^*$.*

(iii) *The open G-orbit in X goes, via the isomorphism $X \cong \operatorname{Spec} \mathbb{C}[\widetilde{\mathcal{O}}]$, to the image of the natural imbedding $\widetilde{\mathcal{O}} \hookrightarrow \operatorname{Spec} \mathbb{C}[\widetilde{\mathcal{O}}]$. The restriction of μ to this orbit goes to the covering map $\widetilde{\mathcal{O}} \twoheadrightarrow \mathcal{O}$.*

(iv) *The Poisson variety X has symplectic singularities and it is a finite union of G-orbits, which are the symplectic leaves of X.*

(v) *If $\mathcal{O}$ is not a Richardson orbit in $\mathfrak{g}$ then X does not have a symplectic resolution.*

Proof. Assume X has an open G-orbit $X_0 \subset X$. Since, any symplectic leaf in X is G-stable, it follows that X_0 is an open subset of the open symplectic leaf. Hence, the restriction of the Poisson structure to X_0 makes the latter a symplectic variety with a transitive G-action. Part (i) is now a consequence of [BrK, Theorem 2.3].

Let $\mathcal{O} := \mu(X_0)$. This is a G-orbit in $\mathfrak{g}^* \cong \mathfrak{g}$, so the map $\mu : X_0 \to \mathcal{O}$ must be a finite covering, cf. Remark 2.2.4. Clearly, we have $\mu(X) = \mu(\overline{X_0}) = \overline{\mathcal{O}}$. This proves (ii). Also, part (i) follows from [BrK, Theorem 2.3].

To complete the proof of (iii), we must show that μ is a finite morphism, equivalently, that $\mathbb{C}[X]$ is a finitely generated module over the subalgebra of $\mathbb{C}[X]$ generated by the subspace $\mathfrak{g} \subset \mathbb{C}[X]$. To this end, let $\mathbb{Z}/(m) \hookrightarrow \mathbb{C}^\times$ be the cyclic group of m-th roots of unity. Then, the $\mathbb{C}^\times$-action on X gives, by restriction, a $\mathbb{Z}/(m)$-action. It follows that $\mathbb{C}[X]$ is a finitely generated module over $A := \mathbb{C}[X/(\mathbb{Z}/(m))] = \oplus_{i \geq 0} \mathbb{C}^{i \cdot m}[X]$, a graded subalgebra in $\mathbb{C}[X]$. We claim next that the algebra A is itself generated by the homogeneous component $\mathbb{C}^m[X]$. that is, by $\mathfrak{g}$ since (i). To see this, note that A is a Poisson subalgebra of $\mathbb{C}[X]$, by (**??**). Furthermore, the vector space $\mathfrak{m} := \oplus_{i>0} \mathbb{C}^{i \cot m}[X]$, is a Poisson ideal of A. Thus, $\mathfrak{m}/\mathfrak{m}^2$ is a finite dimensional graded Lie algebra is a $\mathbb{Z}/(m)$-isotypic component of $\mathbb{C}[X]$. Thanks to Hilbert, it follows that $\mathbb{C}[X]^{(0)} = \oplus_{i \geq 0} \mathbb{C}^{i \cot m}[X]$. is a finitely generated graded algebra and, moreover, each $\mathbb{C}[X]^{(k)}$ is a finitely generated graded $\mathbb{C}[X]^{(0)}$-modu. Since This implies thaf is a finitely generated ideal in that algebra.

Therefore, action on open G-orbit Let $e \in \mathfrak{g}$ be a nonzero nilpotent element. We write $G(e)$ for the G-orbit of e in $\mathfrak{g} \cong \mathfrak{g}^*$, resp. G_e for the centralizer of e in G, and G_e^o for the identity connected component of the group G_e. We have a isomorphism $G(e) \cong G/G^e$, of G-varieties. ☐

Example 4.4.2. Fix a simple Lie algebra $\mathfrak{g}$ and let G be a simply connected group with Lie algebra $\mathfrak{g}$.

Following Brylinski and Kostant [BrK], we fix a subgroup $G_e^o \subset H \subset G_e$ and let $X := (G/H)^{\mathtt{aff}}$ be the affinization of the space G/H. Thus, X is a normal affine G-variety that contains G/H as a Zariski open dense G-orbit. We have a chain of G-equivariant maps $G/H \twoheadrightarrow G/G_e \xrightarrow{\sim} G(e) \hookrightarrow \mathfrak{g}$, where the first map is the natural projection. The composite of the above maps induces, by taking affinizations,

So, taking a pull-back of the canonical symplectic 2-form on $G(e)$ makes G/H a symplectic homogeneous G-variety. The corresponding Poisson bracket on $\mathbb{C}[G/H] = \mathbb{C}[X]$ gives X a structure of Poisson G-variety.

There is a canonical nice $\mathbb{C}^\times$-action on X defined as follows, [BrK]. We choose an $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle \subset \mathfrak{g}$. Associated with that triple, there is an group imbedding $SL_2 \hookrightarrow G$. Let $\gamma : \mathbb{C}^\times \to G$ be the restriction of this imbedding to the diagonal torus $\mathbb{C}^\times \subset SL_2$. One checks that

$\gamma(z)H\gamma(z)^{-1} = H$, for any $z \in \mathbb{C}^\times$. Therefore, the map $gH \mapsto g\gamma(z)H$ gives a well-defined $\mathbb{C}^\times$-action on G/H. The constructed action commutes with the G-action and, moreover, it does not depend on the choice of an $\mathfrak{sl}_2$-triple. Finally, the $\mathbb{C}^\times$-action on G/H gives a grading on the algebra $\mathbb{C}[X] = \mathbb{C}[G/H]$, whence, a $\mathbb{C}^\times$-action on X. One of the main results of [BrK] reads:

Theorem 4.4.3. (i) *The grading $\mathbb{C}[X] = \bigoplus_k \mathbb{C}^k[X]$, satisfies conditions* (**??**)-(**??**) *with $m = 2$.*
(ii) *The Lie algebra $\mathfrak{g}_X = \mathbb{C}^2[X]$ is semisimple.*
(iii) *The map μ^* gives a Lie algebra imbedding $\mathfrak{g} \hookrightarrow \mathfrak{g}_X$.*

A striking observation of Brylinski and Kostant was that, for some choices of the nilpotent $e \in \mathfrak{g}$ and the group H, the imbedding $\mathfrak{g} \hookrightarrow \mathfrak{g}_X$ is *not* a bijection. Thus, the triple $(\mathfrak{g}, e, H)$ gives rise to a semisimple Lie algebra $\mathfrak{g}_X$ that is strictly larger than $\mathfrak{g}$. In [BrK], one can find a complete classification of all such triples $(\mathfrak{g}, e, H)$.

The results of [BrK] has an interesting connection to a conjectural classification of of smooth projective Fano varieties with a contact structure, see [Bea2].

4.5. **A Lagrangian subvariety.** We recall the following standard

Definition. Let Y be a smooth variety with an algebraic symplectic 2-form ω. A locally closed subvariety $\Lambda \subset Y$ is called *Lagrangian* if the tangent space to Λ at any smooth point $\phi \in \Lambda$ is a maximal isotropic subspace of $T_\phi Y$ (the tangent space to M at ϕ) with respect to the symplectic 2-form ω.

Let $\widetilde{X}$ be a smooth symplectic variety with a $\mathbb{C}^\times$-action, X an arbitrary affine $\mathbb{C}^\times$-variety X, and $\pi : \widetilde{X} \to X$ a proper $\mathbb{C}^\times$-equivariant morphism $\pi : \widetilde{X} \to X$ (not necessarily a symplectic resolution). We let $\Lambda := [\pi^{-1}(X^{\mathbb{C}^\times})]_{\mathrm{red}}$ denote the preimage of the $\mathbb{C}^\times$-fixed point set, equipped with reduced scheme structure. Thus, $\Lambda \subset \widetilde{X}$ is a reduced closed subscheme.

Theorem 4.5.1. *Assume that the following holds:*
(i) *The $\mathbb{C}^\times$-action on X ia a contraction to $X^{\mathbb{C}^\times}$, and*
(ii) *The 2-form ω has weight 1 with respect to the $\mathbb{C}^\times$-action, i.e., for any $t \in \mathbb{C}^\times$, we have $t^*(\omega) = t \cdot \omega$.*
Then, each irreducible component of Λ is a Lagrangian subvariety.

Lemma 4.5.2. *The set F is contained in $\pi^{-1}(X^{\mathbb{C}^\times})$, and there is a set-theoretic decomposition $\Lambda = \bigsqcup_{1\leq s\leq r} \Lambda_s$.*

Proof. Since π is a $\mathbb{C}^\times$-equivariant morphism, we have $\pi(\widetilde{X}^{\mathbb{C}^\times}) \subset X^{\mathbb{C}^\times}$. In particular, one has $F \subset \pi^{-1}(X^{\mathbb{C}^\times})$.

Now, fix $\widetilde{z} \in \widetilde{X}$ and let $z = \pi(\widetilde{z}) \in X$. We consider the maps $\mathbb{C}^\times \to X$, $t \mapsto t(z)$, resp. $\mathbb{C}^\times \to \widetilde{X}$, $t \mapsto t(\widetilde{z})$. Assume first that $\widetilde{z} \in \Lambda$. Then, z is a $\mathbb{C}^\times$-fixed point and $t(\widetilde{z}) \in \pi^{-1}(z)$ for any t. It follows, since $\pi^{-1}(z)$ is a complete variety, that the map $t \mapsto t(\widetilde{z})$ extends to a regular map $\mathbb{P}^1 \to \widetilde{X}$. Thus, for any $\widetilde{z} \in \pi^{-1}(X^{\mathbb{C}^\times})$, the limit of $t(\widetilde{z}),\ t \to \infty$, exists and we have $\lim_{t\to\infty} t(\widetilde{z}) \in F$. We conclude that $\Lambda \subset \cup_{1\leq s\leq r} \Lambda_s$.

Next let $\widetilde{z} \in \cup_{1\leq s\leq r} \Lambda_s$, so we have $\lim_{t\to\infty} t(\widetilde{z}) \in F$. It follows that the map $t \mapsto \pi(t(\widetilde{z})) = t(z)$ also has a limit as $t \to \infty$. Therefore, the map $\mathbb{C}^\times \to X,\ t \mapsto t(z)$ extends to the point $t = \infty$. On the other hand, this map extends to the point $t = 0$, thanks to assumption (i) of Theorem 4.5.1. Therefore, we get a regular map $\mathbb{P}^1 \to X$. Such a map must be a constant map, since X is affine. Thus, we must have $\pi(\widetilde{z}) = z \in X^{\mathbb{C}^\times}$. We conclude that $\widetilde{z} \in \Lambda$. The result follows. ☐

Remark 4.5.3. We have shown that the $\mathbb{C}^\times$-action provides a contraction of the variety $\widetilde{X}$ to the fixed point set F.

4.6. Theorem 4.5.1 is clearly a consequence of the following more precise result

Proposition 4.6.1. *Each piece Λ_s is a smooth, connected, locally closed Lagrangian subvariety of $\widetilde{X}$. Furthermore, the closures $\overline{\Lambda}_s,\ s=1,\dots,r$, are precisely the irreducible components of Λ.*

Proof. Fix a connected component F_s and a point $\phi \in F_s$. The tangent space to $\widetilde{X}$ at ϕ has a weight decomposition with respect to the $\mathbb{C}^\times$-action

$$T_\phi \widetilde{X} = \bigoplus\nolimits_{m\in\mathbb{Z}} H_m, \tag{4.6.2}$$

such that $t\in\mathbb{C}^\times$ acts on the direct summand H_m via multiplication by t^m. In particular, we see that $H_0 = T_\phi F$, is the tangent space to the fixed point set F.

Recall that the symplectic 2-form ω on $\widetilde{X}$ has weight $+1$ with respect to the $\mathbb{C}^\times$-action. Hence, a pair of direct summands H_k and H_l are ω-orthogonal unless $k+l=1$; furthermore, the 2-form gives a perfect pairing $\omega:\ H_m\times H_{1-m}\to\mathbb{C}$, for any $m\in\mathbb{Z}$. We see, in particular, that $\bigoplus_{m\leq 0} H_m$ is a Lagrangian subspace in $\bigoplus_{m\in\mathbb{Z}} H_m$.

To complete the proof, pick $z\in\Lambda_s$ such that $\lim\limits_{t\to\infty} t(z)=\phi$. It is clear that, for the curve $t\mapsto t(z)$ to have a limit as $t\to\infty$, the tangent vector to the curve at $t=\infty$ must belong to the span of nonpositive weight subspaces. In other words, we must have

$$\frac{d(t(z))}{dt}\Big|_{t=\infty} \in \bigoplus\nolimits_{m<0} H_m.$$

Since Λ_s is smooth at ϕ, we deduce the equation $T_\phi\Lambda_s = \bigoplus_{m\leq 0} H_m$. It follows, by the above, that $T_\phi\Lambda_s$ is a Lagrangian subspace in $T_\phi\widetilde{X}$, and the first statement of the proposition is proved.

Now, the decomposition of Lemma 4.5.2 presents Λ as a union of irreducible varieties of equal dimensions, and the second statement of the proposition follows. ∎

A similar argument yields the following analogue of Proposition 4.6.1 in the case where the symplectic form has weight 0 rather than weight 1.

Proposition 4.6.3. *Let X be symplectic variety with a $\mathbb{C}^\times$-action that the symplectic 2-form on X is $\mathbb{C}^\times$-invariant, the set $X^{\mathbb{C}^\times}$ is finite, and the $\mathbb{C}^\times$-action on X is a contraction to $X^{\mathbb{C}^\times}$. Let Λ_s be defined by formula* (4.1.1)*, as before.*

With these assumptions, all the statements of Proposition 4.6.1 hold true.

5. Symplectic resolutions

5.1. Recall that any smooth symplectic algebraic manifold carries a natural Poisson structure.

Definition. Let X be an irreducible affine normal Poisson variety. A resolution of singularities $\pi:\widetilde{X}\to X$ is called a *symplectic resolution* of X provided $\widetilde{X}$ is a smooth complex algebraic symplectic manifold (with algebraic symplectic 2-form) such that the pull-back morphism $\pi^*:\mathbb{C}[X]\to\Gamma(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ is a Poisson algebra morphism.

Obviously, any Poisson variety that admits a symplectic resolution has symplectic singularities. A converse does not hold: there are many varieties with symplectic singularities which do not have a symplectic resolution.

5.2. We discuss now several especially important examples of symplectic resolutions.

Example 5.2.1 (**Slodowy slices**). Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\langle e,h,f\rangle\subset\mathfrak{g}$ an $\mathfrak{sl}_2$-triple for a nilpotent element $e\in\mathfrak{g}$. Write $\mathfrak{z}_f$ for the centralizer of f in $\mathfrak{g}$, and $\mathcal{N}$ for the *nilcone*, the subvariety of all nilpotent elements of $\mathfrak{g}$. Slodowy has shown that the intersection $\mathcal{S}_e:=\mathcal{N}\cap(e+\mathfrak{z}_f)$

is reduced, normal, and that there is a $\mathbb{C}^\times$-action on $\mathcal{S}_e$ that contracts $\mathcal{S}_e$ to e, cf. eg. [CG], §3.7 for an exposition.

The variety $\mathcal{S}_e$ is called the *Slodowy slice* for e (the variety $\mathcal{S}_e$ has been known already to Harish-Chandra; it was studied in detail and extensively used by P. Slodowy [Sl]).

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the Killing form, and view $\mathcal{S}_e$ as a subvariety in $\mathfrak{g}^*$. Then, the standard Kirillov-Kostant Poisson structure on $\mathfrak{g}^*$ induces a Poisson structure on $\mathcal{S}_e$. The symplectic leaves in $\mathcal{S}_e$ are obtained by intersecting $e + \mathfrak{z}_f$, an affine space, with the various nilpotent conjugacy classes in $\mathfrak{g}$.

Let $\mathcal{B}$ denote the flag variety for $\mathfrak{g}$, that is, the variety of all Borel subalgebras in $\mathfrak{g}$, and let $T^*\mathcal{B}$ be the cotangent bundle on $\mathcal{B}$. There is a standard resolution of singularities $\pi : T^*\mathcal{B} \to \mathcal{N}$, the *Springer resolution*, cf. eg. [CG, ch. 3]. It is known that $\widetilde{\mathcal{S}}_e := \pi^{-1}(\mathcal{S}_e)$ is a smooth submanifold in $T^*\mathcal{B}$ and the canonical symplectic 2-form on the cotangent bundle restricts to a nondegenerate, hence symplectic, 2-form on $\widetilde{\mathcal{S}}_e$. Moreover, restricting π to $\widetilde{\mathcal{S}}_e$ gives a symplectic resolution $\pi_e : \widetilde{\mathcal{S}}_e \to \mathcal{S}_e$, see [Sl] and also [Gi2], Proposition 2.1.2. The central fiber of that resolution is $\pi_e^{-1}(e) = \mathcal{B}_e$, the fixed point set of the natural action of the element $e \in \mathfrak{g}$ on the flag variety $\mathcal{B}$.

In the (somewhat degenerate) case $e = 0$, we have $\mathcal{S}_e = \mathcal{N}$, and the corresponding symplectic resolution reduces to the Springer resolution itself.

Example 5.2.2 (**Symplectic orbifolds**). Let (V, ω) be a finite dimensional symplectic vector space and $\Gamma \subset Sp(V, \omega)$ a finite subgroup. The orbifold $X := V/\Gamma$ is an affine normal algebraic variety, and the symplectic structure on V induces a Poisson structure on X. Such a variety X may or may not have a symplectic resolution $\widetilde{X} \to X$, in general. This holds, for instance, in the case of *Kleinian singularities*, that is the case where $\Gamma \subset SL_2(\mathbb{C})$ and $X := \mathbb{C}^2/\Gamma$. Then, a symplectic resolution $\pi : \widetilde{X} \to X$ does exist. It is the canonical minimal resultion, see [Kro].

Recall that there is a correspondence, the McKay correspondence, between the (conjugacy classes of) finite subgroups $\Gamma \subset SL_2(\mathbb{C})$ and Dynkin graphs of $\mathbf{A}, \mathbf{D}, \mathbf{E}$ types, cf. [CS]. It turns out that $\mathbb{C}^2/\Gamma$ is isomorphic, as a Poisson variety, to the Slodowy slice $\mathcal{S}_e$, where e is a *subregular* nilpotent in the simple Lie algebra $\mathfrak{g}$ associated with the Dynking diagram of the corresponding type.

Another important example is the case where $\Gamma \subset GL(\mathfrak{h})$ is a complex reflection group and $V := \mathfrak{h} \times \mathfrak{h}^* = T^*\mathfrak{h}$ is the cotangent bundle of the vector space $\mathfrak{h}$ equipped with the canonical symplectic structure of the cotangent bundle. We get a natural imbedding $\Gamma \subset Sp(V)$. One can show that, among all irreducible finite Weyl groups Γ, only those of types $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, have the property that the orbifold $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ admits a symplectic resolution, see [GK], [Go].

In type $\mathbf{A}$, we have $\Gamma = S_n$, the Symmetric group acting diagonally on $\mathbb{C}^n \times \mathbb{C}^n$ (two copies of the permutation representation). Thus, $(\mathbb{C}^n \times \mathbb{C}^n)/S_n = (\mathbb{C}^2)^n/S_n$ is the n-th symmetric power of the plane $\mathbb{C}^2$. The orbifold $(\mathbb{C}^2)^n/S_n$ has a natural resolution of singularities $\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to (\mathbb{C}^2)^n/S_n$, where $\mathrm{Hilb}^n(\mathbb{C}^2)$ stands for the Hilbert scheme of n points in $\mathbb{C}^2$. The map π, called Hilbert-Chow morphism, turns out to be a symplectic resolution, cf. [Na3], §1.4.

Example 5.2.3 (**Quiver varieties**). Let Q be a finite quiver with vertex set I. Let $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I_{\geq 0}$ be a pair of dimension vectors. Nakajima varieties provide, in many cases, examples of a symplectic resolution of the form $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Here, $\theta \in \mathbb{R}^I$ is a 'stability parameter', and we write $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ for the Nakajima variety $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$. For $\theta = 0$, the variety $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ is known to be affine.

6. Poisson deformations.

6.1. Given a smooth symplectic, resp. not necessarily smooth Poisson, variety, one can study its flat deformations in the category of symplectic, resp. Poisson, varieties. The corresponding theory was initiated in [GK] and was further developed by Namikawa in [N1]-[N2]. Note that any deformation of a smooth symplectic variety Y in the category of symplectic varieties induces canonically a deformation of $Y^{\mathtt{aff}}$ in the category of Poisson varieties. For example, let $\pi : \widetilde{X} \to X$ be a symplectic resolution where X is an affine normal variety, as usual. Then, one has $X = \widetilde{X}^{\mathtt{aff}}$ so π is, in fact, the affinization morphism. Thus, we see that any symplectic deformation of $\widetilde{X}$ induces a Poisson deformation of X.

A much deeper connection between Poisson deformations and symplectic resolutions is provided by the following result of Namikawa[N2]:

Theorem 6.1.1. *Let X be an affine variety with* symplectic singularities *equipped with a nice $\mathbb{C}^\times$-action, cf.* (**??**). *Then, X admits a flat Poisson deformation to a* smooth *Poisson variety if and only if X has a symplectic resolution.*

Next, we study deformations of symplectic resolutions.

Fix a $\mathbb{C}^\times$-equivariant symplectic resolution $\pi : \widetilde{X} \to X$, where X is a Poisson variety with a nice $\mathbb{C}^\times$-action. It turns out that the vector space $\mathfrak{H} := H^2(X)$ is the natural parameter space for deformations of the symplectic resolution π. Specifically, define a $\mathbb{C}^*$-action on $\mathfrak{H}$ by $z : h \mapsto z \cdot h$. Then, one has the following result, see [GK, Theorem 1.13]:

Theorem 6.1.2. *Given a symplectic resolution $\pi : \widetilde{X} \to X$ as above, there exists a smooth $\mathbb{C}^*$-variety $\widetilde{\mathfrak{X}}$ and a smooth $\mathbb{C}^*$-equivariant morphism $\boldsymbol{\phi} : \widetilde{\mathfrak{X}} \to \mathfrak{H}$ such that the following holds.*

Put $\mathfrak{X} := \widetilde{\mathfrak{X}}^{\mathtt{aff}}$ and let $\boldsymbol{\phi} = f \circ \boldsymbol{\pi} : \widetilde{\mathfrak{X}} \to \mathfrak{X} \to \mathfrak{H}$ be the canonical factorization, where $\boldsymbol{\pi}$ is the affinization morphism and f is an affine morphism. For $h \in \mathfrak{H}$, put $\widetilde{\mathfrak{X}}_h := \boldsymbol{\phi}^{-1}(h)$, resp. $\mathfrak{X}_h := f^{-1}(h)$. Then, we have:

1. *$\widetilde{\mathfrak{X}}$ is a relative symplectic manifold over $\mathfrak{H}$, i.e., there is a relative 2-form $\omega \in \Gamma(\widetilde{\mathfrak{X}}, \Omega^2_{\widetilde{\mathfrak{X}}/\mathfrak{H}})$ such that, for every $h \in \mathfrak{H}$, the restriction $\omega|_{\widetilde{\mathfrak{X}}_h}$ is a symplectic 2-form on the fiber $\widetilde{\mathfrak{X}}_h$.*
2. *The map f is flat, so the family $\{\mathfrak{X}_h,\ h \in \mathfrak{H}\}$ is a flat family of affine normal Poisson varieties.*
3. *The map $\boldsymbol{\pi}$ is a projective birational morphism such that, for every $h \in \mathfrak{H}$, the map $\boldsymbol{\pi}|_{\widetilde{\mathfrak{X}}_h} : \widetilde{\mathfrak{X}}_h \to \mathfrak{X}_h$ is a symplectic resolution. Moreover, there exists a Zariski open dense subset $\mathfrak{H}^\circ \subset \mathfrak{H}$ such that the map $\boldsymbol{\pi}$ restricts to an isomorphism $\widetilde{\mathfrak{X}}^\circ \xrightarrow{\sim} \mathfrak{X}^\circ$, where $\mathfrak{X}^\circ := f^{-1}(\mathfrak{H}^\circ)$ and $\widetilde{\mathfrak{X}}^\circ := \boldsymbol{\phi}^{-1}(\mathfrak{H}^\circ) = \boldsymbol{\pi}^{-1}(\mathfrak{X}^\circ)$.*
4. *There is a $\mathbb{C}^\times$-equivariant isomorphism $\widetilde{\mathfrak{X}}_0 \cong \widetilde{X}$, of symplectic algebraic varieties, such that the map $\boldsymbol{\pi}|_{\widetilde{\mathfrak{X}}_0} : \widetilde{\mathfrak{X}}_0 \to (\widetilde{\mathfrak{X}}_0)^{\mathtt{aff}} = \mathfrak{X}_0$ gets identified, via the isomorphism, with the map $\pi : \widetilde{X} \to (\widetilde{X})^{\mathtt{aff}} = X$, the symplectic resolution. Thus, one has the following commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathfrak{X}}^\circ \arrow[r, hook] \arrow[d, "\boldsymbol{\pi}", "\cong"'] & \widetilde{\mathfrak{X}} \arrow[d, "\boldsymbol{\pi}"] \arrow[dd, bend right, "\boldsymbol{\phi}"'] & \widetilde{\mathfrak{X}}_0 \arrow[l, hook'] \arrow[r, equal] \arrow[d, "\boldsymbol{\pi}"] & \widetilde{X} \arrow[d, "\pi"] \\
\mathfrak{X}^\circ \arrow[r, hook] \arrow[d, "f"] & \mathfrak{X} \arrow[d, "f"] & \mathfrak{X}_0 \arrow[l, hook'] \arrow[r, equal] \arrow[d] & X \arrow[d] \\
\mathfrak{H}^\circ \arrow[r, hook] & \mathfrak{H} & \{0\} \arrow[l, hook'] \arrow[r, equal] & \{0\}
\end{tikzcd} \tag{6.1.3}$$

We observe that the morphism f, in diagram (6.1.3), being affine the last statement in Theorem (6.1.2)(3) implies the following

Corollary 6.1.4. *The symplectic variety $\widetilde{\mathfrak{X}}_h$ is affine, for any $h \in \mathfrak{H}^\circ$.*

6.2. **Nearby cycles for symplectic resolutions.** Recall that a morphism $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is a smooth connected variety, is called *semismall* (in the sense of Goresky-MacPherson) if one has an equation $\dim(\widetilde{X} \times_X \widetilde{X}) = \dim X$. Note that the set $\widetilde{X} \times_X \widetilde{X}$ may have several irreducible components, so the semismallness requires the dimension of any such component be $\leq \dim X$. The diagonal $X \subset \widetilde{X} \times_X \widetilde{X}$ has dimension $\dim X$, so, for a semismall map π, the diagonal is an irreducible component of $\widetilde{X} \times_X \widetilde{X}$ of maximal dimension.

We will use the techniques of nearby cycles to give an alternative short proof of the following result of Kaledin, [K4].

Theorem 6.2.1. *Let $\pi : \widetilde{X} \to X$ be a symplectic resolution of an affine normal Poisson variety X with a nice $\mathbb{C}^\times$-action. Then, π is a semi-small morphism.*

Proof. Given a smooth connected variety Y, we let $C_Y := \mathbb{C}_Y[\dim Y]$ be a constant sheaf on Y viewed as a complex concentrated in degree $(-\dim Y)$. This degree shift insures that C_Y is a perverse sheaf on Y.

To prove the theorem, we observe that the map π is semismall if and only if $R\pi_* C_{\widetilde{X}}$, a derived direct image, is a perverse sheaf on X. To show the latter, we use the deformation provided by Theorem 6.1.2. It will be more convenient, however, to have a 1-parameter deformation rather than the deformation over the base $\mathfrak{H} = H^2(\widetilde{X})$ that may have dimension > 1, in general.

To construct such a deformation, one chooses a relatively ample line bundle L, on $\widetilde{X}$. Let $c_1(L) \in \mathfrak{H} = H^2(\widetilde{X})$ be the first Chern class of L and let $\mathbb{C} \subset \mathfrak{H}$ denote the image of the imbedding $\mathbb{C} \hookrightarrow \mathfrak{H}$, $z \mapsto z \cdot c_1(L)$. Let $\widetilde{\mathfrak{X}}_{\mathbb{C}}$ be the restriction of the deformation $\mathfrak{X} \to \mathfrak{H}$, of Theorem 6.1.2 to the line $\mathbb{C}$. This way, we obtain a flat 1-parameter deformation $\widetilde{\mathfrak{X}}_{\mathbb{C}} \to \mathbb{C}$, called the *twistor deformation* associated with the line bundle L. This is a symplectic deformation of $\widetilde{X}$ that can also be constructed in a more direct way that does not involve Theorem 6.1.2, see [K6].

Next, we pull back diagram (6.1.3) via the imbedding $\mathbb{C} \hookrightarrow \mathfrak{H}$. This yields a diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{X}}^\circ_{\mathbb{C}} \arrow[r, hook] \arrow[d, "\varpi^\circ"] & \widetilde{\mathfrak{X}}_{\mathbb{C}} \arrow[d, "\varpi"] & \widetilde{X} \arrow[l, hook'] \arrow[d, "\pi"] \\
\mathfrak{X}^\circ_{\mathbb{C}} \arrow[r, hook] \arrow[d, "f"] & \mathfrak{X}_{\mathbb{C}} := \widetilde{\mathfrak{X}}^{\mathtt{aff}}_{\mathbb{C}} \arrow[d, "f"] & X \arrow[l, hook'] \arrow[d, "f"] \\
\mathbb{C}^\times \arrow[r, hook] & \mathbb{C} & \{o\} \arrow[l, hook']
\end{tikzcd}$$

Thus, we have factored the twistor deformation as a composition $f \circ \varpi : \widetilde{\mathfrak{X}}_{\mathbb{C}} \to \mathfrak{X}_{\mathbb{C}} \to \mathbb{C}$.

Let ψ_f, resp. $\psi_{f \circ \varpi}$, denote the nearby cycles functor associated with the function f, resp. $f \circ \varpi$. This is a functor that sends constructible complexes on the generic fiber to constructible complexes on the special fiber.

First, we note that the map $f \circ \varpi$, the twistor deformation, is a smooth morphism. This implies an isomorphism $\psi_{f \circ \varpi}(\mathbb{C}_{\widetilde{\mathfrak{X}}^\circ_{\mathbb{C}}}) = \mathbb{C}_{\widetilde{X}}$. Further, it follows from Theorem 6.1.2 that the map $\varpi^\circ : \widetilde{\mathfrak{X}}^\circ_{\mathbb{C}} \to \mathfrak{X}^\circ_{\mathbb{C}}$, in the above diagram, is an isomorphism. Hence, $\mathfrak{X}^\circ_{\mathbb{C}}$ is a smooth variety and we have $\varpi^\circ_* C_{\widetilde{\mathfrak{X}}^\circ_{\mathbb{C}}} = C_{\mathfrak{X}^\circ_{\mathbb{C}}}$. Thus, proper base change for nearby cycles yields

$$\psi_f(C_{\mathfrak{X}^\circ_{\mathbb{C}}}) = \psi_f(\varpi^\circ_* C_{\widetilde{\mathfrak{X}}^\circ_{\mathbb{C}}}) = \varpi_* \psi_{f \circ \varpi}(C_{\widetilde{\mathfrak{X}}^\circ_{\mathbb{C}}}) = \pi_* C_{\widetilde{X}}.$$

Finally, it is known that the nearby cycles functor sends perverse sheaves to perverse sheaves, see e.g. [GM, §6.1]. It follows that $\pi_* C_{\widetilde{X}} \cong \psi_f(C_{\mathfrak{X}^\circ_{\mathbb{C}}})$ is a perverse sheaf on X, as desired. $\square$

Closely related to the above argument, is the following result.

Corollary 6.2.2. *Let $\varpi : \widetilde{\mathfrak{X}}_{\mathbb{C}} \to \mathfrak{X}_{\mathbb{C}}$ be a twistor deformation of the symplectic resolution $\pi : \widetilde{X} \to X$. Then, we have $R\varpi_* C_{\widetilde{\mathfrak{X}}_{\mathbb{C}}} = \mathrm{IC}_{\mathfrak{X}_{\mathbb{C}}}$, the intersection cohomology complex of the singular variety $\mathfrak{X}_{\mathbb{C}}$. In particular, for any $x \in X$, one has an isomorphism*

$$H^\bullet(X_x) \cong \mathscr{H}_x^{\bullet - \dim X - 1}(\mathrm{IC}_{\mathfrak{X}_{\mathbb{C}}}).$$

Proof. We have shown that the map π is semismall. Then, it follows directly from the dimension bounds involved in the definitions of small and semismall maps that the map $\varpi : \widetilde{\mathfrak{X}}_{\mathbb{C}} \to \mathfrak{X}_{\mathbb{C}}$ is small. This implies the result. $\square$

Remark 6.2.3. An isomorphism $R\varpi_* C_{\widetilde{\mathfrak{X}}_{\mathbb{C}}} = \mathrm{IC}_{\mathfrak{X}_{\mathbb{C}}}$ is equivalent to saying that the morphism ϖ is *small* (in the sense of Goresky-MacPherson).

7. Purity

7.1. We keep the setup of the previous subsection. The main result of this subsection, Theorem 7.1.1 below, says that the cohomology groups $H^\bullet(\widetilde{\mathfrak{X}}_h)$ and $H^\bullet(\widetilde{\mathfrak{X}}_0)$ are *canonically* isomorphic to each other, for every point $h \in \mathfrak{H}$. The isomorphism is, in fact, provided by the Gauss-Manin connection.

Let $[\omega_h] \in H^2(\widetilde{\mathfrak{X}}_h)$ be the de Rham cohomology class of the symplectic 2-form ω_h. Thanks to the canonical isomorphism $H^\bullet(\widetilde{\mathfrak{X}}_h) \cong H^\bullet(\widetilde{\mathfrak{X}}_0)$, the assignment $h \mapsto [\omega_h] \in$ gives a well defined map *per* : $\mathfrak{H} \to H^2(\widetilde{\mathfrak{X}}_0)$, called the *period map*. Note that the 2-form ω_0 is exact since one has an equation $\omega = \frac{1}{m} d(i_{\mathsf{eu}}\omega_0)$, where eu is the Euler vector field induced by the $\mathbb{C}^\times$-action on $\widetilde{X}$, cf. Example 2.2.7. It follows that we have *per*$(0) = [\omega_0] = 0$. Moreover, it is not difficult to show that the period map is in fact equal to the identity map $H^2(\widetilde{\mathfrak{X}}_0) = \mathfrak{H} \to H^2(\widetilde{\mathfrak{X}}_0)$. This implies that the deformation $\widetilde{\mathfrak{X}} \to \mathfrak{H}$ is semi-universal, in the sense of deformation theory.

Theorem 7.1.1. *The sheaf $R^k\phi_*\mathbb{C}_{\widetilde{\mathfrak{X}}}$ is a constant sheaf on $\mathfrak{H}$ and the following restriction maps*

$$H^k(\widetilde{X}) \xrightarrow{\cong} H^k(\widetilde{X}_o), \qquad H^k(\widetilde{\mathfrak{X}}) \xrightarrow{\cong} H^k(\widetilde{\mathfrak{X}}_h), \qquad k \geq 0, \tag{7.1.2}$$

are isomorphisms, for all $h \in \mathfrak{H}$. Moreover, each of the cohomology groups above is pure of weight k.

7.2. **Proof of Theorem 7.1.1.** Associated with any locally closed subvariety $S \subset \mathfrak{H}$, there are various objects $\widetilde{\mathfrak{X}}_S := S \times_{\mathfrak{H}} \widetilde{\mathfrak{X}} = \phi^{-1}(S)$, resp. $\mathfrak{X}_S := S \times_{\mathfrak{H}} \mathfrak{X} = f^{-1}(S)$ and $\phi_S := \phi|_{\widetilde{\mathfrak{X}}_S}$, etc., obtained from the corresponding objects of diagram (6.1.3) by base change via the imbedding $S \hookrightarrow \mathfrak{H}$. In the special case where $S = \{h\}$ is a one-point set, we have $\widetilde{\mathfrak{X}}_{\{h\}} = \widetilde{\mathfrak{X}}_h$.

Let $S \subset \mathfrak{H}$ be a closed $\mathbb{C}^\times$-stable subvariety. Thus, we have $\mathbb{C}^\times$-equivariant maps $\widetilde{\mathfrak{X}}_S \to \mathfrak{X}_S \to S$, where the first map is projective morphism and the composite of the two maps is a smooth morphism. Note that S contains zero, moreover, we have $S^{\mathbb{C}^\times} = \{0\}$. It follows that $\mathfrak{X}_S$ contains X as a closed subvariety and, we have $(\mathfrak{X}_S)^{\mathbb{C}^\times} = X^{\mathbb{C}^\times} = \{o\}$. We deduce that $\widetilde{X} \subset \widetilde{\mathfrak{X}}_S$ and, moreover, $(\widetilde{\mathfrak{X}}_S)^{\mathbb{C}^\times} = \widetilde{X}^{\mathbb{C}^\times} = \widetilde{X}_o^{\mathbb{C}^\times}$ is the $\mathbb{C}^\times$-fixed point set in the central fiber. Hence, this fixed point set is a projective variety.

The above implies, thanks to a well known result due to Springer [Sp, Corollary 1], that the natural restriction map $H^\bullet(\widetilde{\mathfrak{X}}_S) \to H^\bullet(\widetilde{X}_o^{\mathbb{C}^\times})$ is an isomorphism. Combining this with similar results in the special cases where $S = \mathfrak{H}$ ansd $S = 0$, respectively, we deduce that each of the restriction maps below must be an isomorphism as well:

$$H^\bullet(\widetilde{\mathfrak{X}}) \xrightarrow{\cong} H^\bullet(\widetilde{\mathfrak{X}}_S) \xrightarrow{\cong} H^\bullet(\widetilde{\mathfrak{X}}_0) = H^\bullet(\widetilde{X}) \xrightarrow{\cong} H^\bullet(\widetilde{X}_o) \xrightarrow{\cong} H^\bullet(\widetilde{X}_o^{\mathbb{C}^\times}). \tag{7.2.1}$$

In particular, this yields the first isomorphism in (7.1.2) and also the second isomorphism in (7.1.2) in the special case $h = 0$.

Further, following Springer we observe that, for any $k \geq 0$, all the weights in the cohomology group $H^k(\widetilde{X})$ are $\geq k$, since $\widetilde{X}$ is a smooth (but not compact) variety. On the other hand, the group $H^k(\widetilde{X}_o)$ has weights $\leq k$ since $\widetilde{X}_o$, the central fiber, is a (typically singular) projective variety. Thus, the isomorphism $H^k(\widetilde{X}) \cong H^k(\widetilde{X}_o)$ in (7.2.1) forces all the cohomology groups which appear in (7.2.1) to be pure of weight k, see [Sp, Theorem 1].

Next, we prove that the cohomology group $H^k(\widetilde{\mathfrak{X}}_h)$ is pure of weight k, for *any* $h \neq 0$. So, fix $h \in \mathfrak{H} \smallsetminus \{0\}$ and put $S := \mathbb{C} \cdot h \subset \mathfrak{H}$, the line spanned by h. Associated with S, we have a closed $\mathbb{C}^\times$-stable subset $\widetilde{\mathfrak{X}}_S = \phi^{-1}(\mathbb{C} \cdot h)$, of $\widetilde{\mathfrak{X}}$. Since $S = \mathbb{C}^\times \cdot h \sqcup \{0\}$, there is a decomposition $\widetilde{\mathfrak{X}}_S = \phi^{-1}(S \smallsetminus \{0\}) \sqcup \widetilde{\mathfrak{X}}_0$, where $\phi^{-1}(S \smallsetminus \{0\}) = \widetilde{\mathfrak{X}}_S \smallsetminus \widetilde{\mathfrak{X}}_0$ is a $\mathbb{C}^\times$-stable open subset of $\widetilde{\mathfrak{X}}_S$. It is clear that the $\mathbb{C}^*$-action on $\widetilde{\mathfrak{X}}_S^\circ$ provides a $\mathbb{C}^*$-equivariant isomorphism

$$\mathbb{C}^* \times \widetilde{\mathfrak{X}}_h \xrightarrow{\sim} \phi^{-1}(\mathbb{C}^\times \cdot h), \quad z \times x \mapsto z(x).$$

Thus, we have a diagram

$$\widetilde{\mathfrak{X}}_0 \xhookrightarrow{\ i\ } \widetilde{\mathfrak{X}}_S \xhookleftarrow{\ j\ } \widetilde{\mathfrak{X}} \smallsetminus \widetilde{\mathfrak{X}}_0 \;=\!=\!=\; \mathbb{C}^* \times \widetilde{\mathfrak{X}}_h, \tag{7.2.2}$$

where i and j are closed and open imbeddings, respectively.

It is well known that the cohomology groups $H^0(\mathbb{C}^\times) = \mathbb{C}(0)$ and $H^1(\mathbb{C}^\times) = \mathbb{C}(2)$ are 1-dimensional vector spaces which have weights 0 and 2, respectively. Hence, one has the following Künneth decomposition

$$H^\bullet(\widetilde{\mathfrak{X}} \smallsetminus \widetilde{\mathfrak{X}}_0) \cong [H^0(\mathbb{C}^\times) \otimes H^\bullet(\widetilde{\mathfrak{X}}_h)] \ \bigoplus \ [H^1(\mathbb{C}^\times) \otimes H^{\bullet-1}(\widetilde{\mathfrak{X}}_h)] \ = \ H^\bullet(\widetilde{\mathfrak{X}}_h) \bigoplus H^{\bullet-1}(\widetilde{\mathfrak{X}}_h)(2).$$

Associated with diagram (7.2.2), there is a standard long exact sequence of cohomology. Using the Künneth decomposition, the long exact sequence takes the following form, where [1] denotes the connecting homomorphism:

$$\ldots \to\ H^{k-1}(\widetilde{\mathfrak{X}}_0)(1) \xrightarrow{i_!} H^k(\widetilde{\mathfrak{X}}_S) \xrightarrow{j^*} \ H^k(\widetilde{\mathfrak{X}}_h) \bigoplus H^{k-1}(\widetilde{\mathfrak{X}}_h)(2) \xrightarrow{[1]} H^{k+1}(\widetilde{\mathfrak{X}}_0)\ \to \ldots \tag{7.2.3}$$

It turns out that using the weight filtration on the cohomology of algebraic varieties the maps in (7.2.3) can be described quite explicitly. In more detail, for $w \in \mathbb{Z}$, let $\operatorname{gr}_w H^\bullet(-)$ denote an associated graded term of weight w in the weight filtration on the cohomology. Applying the functor $\operatorname{gr}_w(-)$, which is an exact functor, to (7.2.3), one obtains an exact sequence of spaces of weight w. Thanks to the purity result proved earlier, we know that $\operatorname{gr}_w H^n(\widetilde{\mathfrak{X}}_0) = \operatorname{gr}_w H^n(\widetilde{\mathfrak{X}}_S) = 0$, whenever $w \neq n$. Hence, for any $w \geq k+2$, the fragment of the resulting exact sequence of spaces of weight w that corresponds to (7.2.3) reads

$$\ldots \to\ 0 \xrightarrow{i_!} 0 \xrightarrow{j^*} \operatorname{gr}_w H^k(\widetilde{\mathfrak{X}}_h) \bigoplus [\operatorname{gr}_{w-2} H^{k-1}(\widetilde{\mathfrak{X}}_h)](2) \xrightarrow{[1]} 0\ \to \ldots \tag{7.2.4}$$

From (7.2.4), we see that $\operatorname{gr}_{w-2} H^{k-1}(\widetilde{\mathfrak{X}}_h) = 0$. It follows that the group $\operatorname{gr}_w H^k(\widetilde{\mathfrak{X}}_h)$ vanishes for any pair of integers w, k, such that $w > k$. On the other hand, since $\widetilde{\mathfrak{X}}_h$ is smooth, we also have $\operatorname{gr}_w H^k(\widetilde{\mathfrak{X}}_h) = 0$ for all $w < k$. Thus, we conclude that, for each k, the group $H^k(\widetilde{\mathfrak{X}}_h)$ is pure of weight k.

The purity implies that the long exact sequence (7.2.3) breaks up into a direct sum $\oplus_{w\in\mathbb{Z}} E^{(w)}$, of long exact sequences $E^{(w)},\ w \in \mathbb{Z}$, such that, for any w, all terms in $E^{(w)}$ are pure of weight w. Furthermore, one finds that each of these long exact sequences actually splits further into length two exact sequences. Specifically, for $k \in \mathbb{Z}$, the long exact sequence $E^{(k)}$ reduces, effectively, to the following pair of isomorphisms:

$$\jmath^* : H^k(\widetilde{\mathfrak{X}}_S) \xrightarrow{\sim} H^k(\widetilde{\mathfrak{X}}_h) \quad \text{and} \quad \imath_! : H^{k-1}(\widetilde{\mathfrak{X}}_0)(2) \xrightarrow{\sim} H^{k+1}(\widetilde{\mathfrak{X}}_S). \tag{7.2.5}$$

Here, the isomorphism $\jmath^*$ comes from the map j^*, in (7.2.3), followed by the first projection.

Now, we prove that, for h as above, the second map in (7.1.2) is an isomorphism. To this end, we consider the following composition $H^k(\widetilde{\mathfrak{X}}) \to H^k(\widetilde{\mathfrak{X}}_S) \to H^k(\widetilde{\mathfrak{X}}_h)$, of two restriction maps. The first of the maps is an isomorphism thanks to (7.2.1). The second map is an isomorphism, by (7.2.5). Hence, the composite map $H^k(\widetilde{\mathfrak{X}}) \to H^k(\widetilde{\mathfrak{X}}_h)$ is an isomorphism, proving (7.1.2).

It remains to show that the cohomology sheaves $R^k\phi_*\mathbb{C}_{\widetilde{\mathfrak{X}}}$ are constant sheaves on $\mathfrak{H}$. To this end, we need to introduce some notation. Given a closed imbedding $\imath : Y \hookrightarrow Z$, of algebraic varieties, let $\imath^! : D^b_{\text{constr}}(Z) \to D^b_{\text{constr}}(Y)$ denote the functor of *derived* restriction with supports in Y, between the corresponding constructible bounded derived categories. Further, for $h \in \mathfrak{H}$, let $i_h : \{h\} \hookrightarrow \mathfrak{H}$, resp. $\widetilde{i}_h : \widetilde{\mathfrak{X}}_h \hookrightarrow \widetilde{\mathfrak{X}}$, denote the corresponding closed imbedding and $i_h^!$, resp. $\widetilde{i}_h^!$, the derived restriction functor. Finally, we let $p : \widetilde{\mathfrak{X}} \times \mathfrak{H} \twoheadrightarrow \mathfrak{H}$ be the second projection and define a map $\varepsilon : \widetilde{\mathfrak{X}} \hookrightarrow \widetilde{\mathfrak{X}} \times \mathfrak{H}$ by the assignment $x \mapsto (x, \phi(x))$. Thus, ε is a closed embedding via the graph of ϕ, so one has a factorization $\phi = p \circ \varepsilon$.

Now, we begin the proof by noting that each cohomology sheaf $R^kp_*\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}}$ is a constant sheaf, by the Künneth formula. Next, we observe that there is a canonical morphism

$$u : R^\bullet p_*\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}} \longrightarrow R^\bullet p_*(\varepsilon_*\varepsilon^*\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}}) = R^\bullet\phi_*\mathbb{C}_{\widetilde{\mathfrak{X}}}.$$

Thus, we would be done provided we can prove that the morphism u is, in fact, an isomorphism. We will prove this by showing that, for every $h \in \mathfrak{H}$, the morphism $i_h^!(u) : i_h^!(R^\bullet p_*\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}}) \to i_h^!(R^\bullet\phi_*\mathbb{C}_{\widetilde{\mathfrak{X}}})$, induced by u, is an isomorphism. This is known to be sufficient to conclude that u is an isomorphism,

The argument below involves the following diagram, where $\phi_h := \phi|_{\widetilde{\mathfrak{X}}_h}$ and p_h stands for a constant map,

$$\begin{tikzcd}
\widetilde{\mathfrak{X}}_h \arrow[r, "\varepsilon|_{\widetilde{\mathfrak{X}}_h}"'] \arrow[d, hook, "\widetilde{i}_h"] \arrow[rr, bend left, "\phi_h"] & \widetilde{\mathfrak{X}} \times \{h\} \arrow[r, "p_h"'] \arrow[d, hook, "\mathrm{Id}\times i_h"] & \{h\} \arrow[d, hook, "i_h"] \\
\widetilde{\mathfrak{X}} \arrow[r, "\varepsilon"] \arrow[rr, bend right, "\phi"'] & \widetilde{\mathfrak{X}} \times \mathfrak{H} \arrow[r, "p"] & \mathfrak{H}.
\end{tikzcd} \tag{7.2.6}$$

It is clear that all commutative squares in the diagram are cartesian. Also, the map ϕ is a smooth morphism, so $\widetilde{\mathfrak{X}}_h = \phi^{-1}(h)$ is a smooth subvariety in $\widetilde{\mathfrak{X}}$ of codimension $n := \dim \mathfrak{H}$. Therefore, applying proper base change to the above diagram one gets the following canonical isomorphisms:

$$i_h^!(R^\bullet p_*\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}}) = (R^\bullet p_h)_*(\mathrm{Id}\times i_h)^!\mathbb{C}_{\widetilde{\mathfrak{X}}\times\mathfrak{H}} = (R^\bullet p_h)_*(\mathbb{C}_{\widetilde{\mathfrak{X}}}[2n]) = H^{\bullet+2n}(\widetilde{\mathfrak{X}});$$
$$i_h^!(R^\bullet\phi_*\mathbb{C}_{\widetilde{\mathfrak{X}}}) = (R^\bullet\phi_h)_*(\widetilde{i}_h^!\mathbb{C}_{\widetilde{\mathfrak{X}}}) = (R^\bullet\phi_h)_*(\mathbb{C}_{\widetilde{\mathfrak{X}}_h}[2n]) = H^{\bullet+2n}(\widetilde{\mathfrak{X}}_h).$$

Thus, the morphism $i_h^!(u)$ goes, via base change, to a morphism $H^{\bullet+n}(\widetilde{\mathfrak{X}}) \to H^{\bullet+n}(\widetilde{\mathfrak{X}}_h)$. One can check that the latter morphism is the restriction morphism induced by the imbedding $\widetilde{i}_h : \widetilde{\mathfrak{X}}_h \hookrightarrow \widetilde{\mathfrak{X}}$. We have proved earlier, cf. (7.1.2), that the restriction morphism in question is an isomorphism, for any $h \in \mathfrak{H}$, cf. (7.1.2). It follows that the morphism u is an isomorphism, completing the proof of the theorem.

8. TILTING GENERATORS

8.1. Given an associative algebra B, let B^{op} be the *opposite* algebra, and B-mod, resp. B-bimod, be the category of *finitely generated* left B-modules, resp. finitely generated B-bimodules. If B has a $\mathbb{Z}$-grading then one may also consider categories B-grmod and B-grbimod, of finitely generated $\mathbb{Z}$-graded B-modules and B-bimodules, respectively.

Let $D^b_{\mathrm{coh}}(Y)$ be the bounded derived category of coherent sheaves on a scheme Y. If Y is affine, we will often make no distinction beteween the equivalent categories $\mathrm{Coh}(Y)$ and $\mathbb{C}[Y]$-mod, resp. $D^b_{\mathrm{coh}}(Y)$ and $D^b(\mathbb{C}[Y]\text{-mod})$.

In the case where the variety Y has a $\mathbb{C}^\times$-action, one can also consider a triangulated category $D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(Y)$, the bounded derived category of $\mathbb{C}^\times$-equivariant complexes of $\mathcal{O}_Y$-modules with coherent cohomology sheaves. For each $n \in \mathbb{Z}$, one has the character $\chi^n : z \mapsto z^n$, of the group $\mathbb{C}^\times$.

Let $\mathscr{T}$ be a $\mathbb{C}^\times$-equivariant coherent sheaf. Then, one defines $\chi^n \otimes \mathscr{T}$ to be the same coherent sheaf as $\mathscr{T}$ but equipped with a new $\mathbb{C}^\times$-equivariant structure obtained from the one on $\mathscr{T}$ by twisting by the character χ^n. Below, Hom always stands for $\mathrm{Hom}_{D^b_{\mathrm{coh}}(Y)}$. With this convention, for $\mathscr{T}$ as above, one has a canonical isomorphism

$$\mathrm{Hom}(\mathscr{T}, \mathscr{T}) = \bigoplus\nolimits_{n\in\mathbb{Z}} \mathrm{Hom}_{D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(Y)}(\mathscr{T}, \chi^n \otimes \mathscr{T}).$$

This isomorphism makes the left hand side a $\mathbb{Z}$-graded algebra. Let $A_{\mathscr{T}} := [\mathrm{Hom}(\mathscr{T}, \mathscr{T})]^{op}$ denote the opposite algebra.

The following important and nontrivial result was proved by Kaledin [K5] using the theory of 'Fedosov quantization' over fields of positive characteristic, developed by Bezrukavnikov and Kaledin [BK].

Theorem 8.1.1 (Kaledin, [K5])**.** *There exists a locally free $\mathbb{C}^\times$-equivariant sheaf $\mathscr{T}$, on $\widetilde{X}$, such that:*

(i) *We have $\mathrm{Ext}^i(\mathscr{T}, \mathscr{T}) = 0$ for all $i > 0$;*

(ii) *The functor $R\,\mathrm{Hom}(\mathscr{T}, -)$ yields an equivalence $D^b_{\mathrm{coh}}(\widetilde{X}) \xrightarrow{\sim} D^b(A_{\mathscr{T}}\text{-mod})$, of triangulated categories.* □

The object $\mathscr{T}$ satisfying conditions (i)-(ii) of the theorem is usually referred to as a *tilting generator* of the category $D^b_{\mathrm{coh}}(\widetilde{X})$.

Let $\mathscr{T}$ be a tilting generator. One has natural graded algebra imbeddings $\mathbb{C}[X] \hookrightarrow \mathbb{C}[\widetilde{X}] \hookrightarrow A_{\mathscr{T}}$, where the first imbedding is a pull-back via the symplectic resolution $\pi : \widetilde{X} \to X$. We will identify the algebra $\mathbb{C}[X]$ with its image under the composite imbedding. This makes $\mathbb{C}[X]$ a *central* graded subalgebra of A such that $A_{\mathscr{T}}$ is a finitely generated graded $\mathbb{C}[X]$-module (since the map π is proper). In particular, $A_{\mathscr{T}}$ is finite over its center and (both left and right) noetherian.

Remark 8.1.2. The equivalence of Theorem 8.1.1 has an equivariant analogue, a triangulated equivalence $D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(\widetilde{X}) \xrightarrow{\sim} D^b(A_{\mathscr{T}}\text{-grmod})$ provided by the same functor $R\,\mathrm{Hom}(\mathscr{T}, -)$. Furthermore, the equivalences in question are, in fact, equivalences of *module categories* over monoidal categories $D^b_{\mathrm{coh}}(X)$, resp. $D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(X)$, where the monoidal structure is given by the derived tensor product $\mathscr{F} \times \mathscr{G} \mapsto \mathscr{F} \overset{L}{\otimes}_{\mathcal{O}_X} \mathscr{G}$. ◊

8.2. Let $A_{\mathscr{T}} = \oplus_{i\in\mathbb{Z}} A_i$ be the grading on $A_{\mathscr{T}}$. The algebra $\mathbb{C}[X]$ being nonnegatively graded, it follows that each homogeneous component A_i is finite dimensional. We remark that the algebra $A_{\mathscr{T}}$ may have a finite number of homogeneous components of negative degrees, in general.

Proposition 8.2.1. (i) *The algebra $A_{\mathscr{T}}$ has finite global dimension, i.e. any $A_{\mathscr{T}}$-module has a finite projective resolution.*

(ii) *There exists a collection $e_1, \ldots, e_n \in A_{\mathscr{T}}$, of homogeneous orthogonal idempotents of degree zero, such that any indecomposable graded $A_{\mathscr{T}}$-module which is projective as an (ungraded) $A_{\mathscr{T}}$-module is isomorphic, up to a grading shift, to a module of the form $A_{\mathscr{T}} e_i$,*

(iii) *Any finitely generated graded $A_{\mathscr{T}}$-bimodule which is projective as an $(A_{\mathscr{T}} \otimes A_{\mathscr{T}}^{op})$-module is isomorphic (up to grading shifts) to a finite direct sum of modules of the form $A_{\mathscr{T}} e_i \boxtimes e_j A_{\mathscr{T}}$.*

(iv) *The algebra $A_{\mathscr{T}}$ has a homogeneous symmetric nondegenerate $\mathbb{C}[X]$-linear trace $\tau : A_{\mathscr{T}} \to \mathbb{C}[X]$.*

Proof. Part (i) follows from (and is equivalent to) the formal smoothness of the scheme $\widetilde{X}$ using a general criterion of formal smoothness due to Grothendieck.

To prove (ii), put $A = A_{\mathscr{T}}$ and write $I \subset \mathbb{C}[X]$ for the augmentation ideal, so one has $\mathbb{C}[X]/I = \mathbb{C}$. For each $k \geq 1$, let $(I^k) \subset A$ be the ideal in A generated by the k-th power of I. This is a graded ideal of finite codimension in A so $A/(I^k)$ is a finite dimensional graded $\mathbb{C}$-algebra. The grading gives a $\mathbb{C}^\times$-action on this algebra by algebra automorphisms. The Jacobson radical of the algebra $A/(I^k)$, being the maximal nilpotent ideal, is therefore stable under the $\mathbb{C}^\times$-action.

Let $\overline{A}$ be the quotient of $A/(I)$ by its Jacobson radical. Thus, $\overline{A}$ is a semisimple finite dimensional graded algebra and we have $\overline{A} = A/J$ where J is the kernel of the composite map $A \twoheadrightarrow A/(I) \twoheadrightarrow \overline{A}$.

By Wedderburn theory, the algebra $\overline{A}$ is isomorphic to a direct sum of simple algebras of the form $\operatorname{End}_{\mathbb{C}} V$, where V is a finite dimensional vector space. Since the number of two-sided ideals in $\overline{A}$ is finite, it follows that any such ideal is $\mathbb{C}^\times$-stable and the decomposition of $\overline{A}$ into simple algebras respects the $\mathbb{C}^\times$-action. Further, any automorphism of the algebra $\operatorname{End}_{\mathbb{C}} V$ is known to be inner. It follows that the $\mathbb{C}^\times$-action on $\operatorname{End}_{\mathbb{C}} V$ comes from a $\mathbb{C}^\times$-action on the vector space V itself. We decompose V into a direct sum of 1-dimensional weight spaces. Then, the projections to these weight spaces provide a complete set of orthogonal minimal idempotents of the algebra $\operatorname{End}_{\mathbb{C}} V$ which are homogeneous of *degree zero*. We conclude that the algebra $\overline{A}$ also has a complete set, say $\bar{e}_1, \ldots, \bar{e}_n$, of homogeneous orthogonal minimal idempotents of degree zero. Thus, $\overline{A}\bar{e}_i$ is a simple $\overline{A}$-module which we may (and will) view as an A-module via the isomorphism $\overline{A} = A/J$.

Now, choose $k \gg 0$ so that the ideal (I^k) has no homogeneous components of degrees ≤ 0. The semisimple algebra $\overline{A}$ is a quotient of the algebra $A/(I^k)$ by a nilpotent ideal and the standard construction of lifting of idempotents shows that one can lift the idempotents $\bar{e}_1, \ldots, \bar{e}_n$ to orthogonal idempotents $e_1^{(k)}, \ldots, e_n^{(k)} \in A/(I^k)$ which are homogeneous of degree zero again. For each $i = 1, \ldots, n$, let e_i be an arbitrary lift of $e_i^{(k)}$ to a degree zero homogeneous element of A. Then, for any i, j, we have that $e_i e_j - \delta_{i,j} \cdot e_i$ is a degree zero homogeneous element of the ideal (I^k), hence, this element must be equal to zero. We conclude that $e_1, \ldots, e_n \in A$ are degree zero orthogonal idempotents which lift $\bar{e}_1, \ldots, \bar{e}_n \in \overline{A}$. Thus, $P_i := Ae_i$ is an indecomposable graded A-module and it is a projective cover of the simple A-module $\overline{A}\bar{e}_i$.

Next, let P be a finitely generated graded A-module. Then, $P/J \cdot P$ is a finite dimensional graded $\overline{A}$-module. Hence, there is an isomorphism $P/J \cdot P \cong \oplus_{i,r} \overline{A}\bar{e}_i[m_{i,r}]$, where $m_{i,r}$ are some integers and $(-)[m]$ denotes grading shift by m. Let $\bar{p}_{i,r}$ be an element of $P/J \cdot P$ corresponding to $e_i[m_{i,r}]$ under this isomorphism and let $p_{i,r} \in P$ be a homogeneous lift of $\bar{p}_{i,r}$. Thus, $\deg p_{i,r} = m_{i,r}$ and replacing $p_{i,r}$ by $e_i p_{i,r}$, if necessary, we may assume in addition that, for any i, j, r, one has $e_i p_{j,r} = \delta_{i,j} p_{j,r}$.

We claim that the elements $p_{i,r}$ generate P. To see this, let $u_1, \ldots, u_\ell$ be a set of homogeneous generators of P and let $N := \max(\deg u_1, \ldots, \deg u_\ell)$. Further, we consider, for each $k \geq 1$, an $A/(I^k)$-module $P/I^k \cdot P$. Since $P/J \cdot P$ is a quotient of $P/I^k \cdot P$ by the ideal $J/(I^k)$, the classes $p_{i,r} \pmod{I^k \cdot P}$ generate $P/I^k \cdot P$ by the Nakayama lemma. Therefore, there are homogeneous elements $a^{(k)}_{i,r,s} \in A$ such that in $P/I^k\cdot P$ we have

$$u_s \,(\mathrm{mod}\, I^k\cdot P) = \sum_{i,r,s} a^{(k)}_{i,r,s} p_{i,r} \,(\mathrm{mod}\, I^k\cdot P). \tag{8.2.2}$$

It is clear that we may assume without loss of generality that $\deg a_{i,r,s} = \deg u_s - m_{i,r}$. On the other hand, we may choose k large enough so that the module $I^k\cdot P$ has no nonzero homogeneous components in degrees $< N$. Then, equation (8.2.2) yields $u_s = \sum_{i,r,s} a^{(k)}_{i,r,s} p_{i,r}$. It follows that the elements $p_{i,r}$ generate P, as claimed. Thus, the map

$$f :\ \oplus_{i,r}\, P_i[m_{i,r}] = \oplus_{i,r}\, Ae_i[m_{i,r}] \to P, \quad \sum_{i,r} \alpha_{i,r} e_i \mapsto \sum_{i,r} \alpha_{i,r} p_{i,r}$$

gives a well defined and surjective morphism of graded A-modules such that the induced map $\widetilde{P}/J\cdot\widetilde{P} \xrightarrow{\sim} P/J\cdot P$ is an isomorphism.

To complete the proof of part (ii), assume now that the graded A-module P is projective as an A-module (with the grading disregarded). Then the kernel K, of the morphism f, splits off as a direct summand. Therefore, K is projective, furthermore, we have

$$K/J\cdot K \ = \ \mathrm{Ker}[\widetilde{P}/J\cdot\widetilde{P} \xrightarrow{\sim} P/J\cdot P] \ = \ 0.$$

It follows by Nakayama that $K/I^kK = 0$ for any $k \geq 1$. This implies that all homogeneous components of K vanish, so f is an isomorphism. Thus, we have proved that any finitely generated graded A-module which is projective as an A-module is isomorphic to a finite direct sum of modules of the form $P_i[m]$. Part (ii) follows. Also, a similar argument in the case of the algebra $A\otimes A^{op}$ yields (iii).

Finally, the statement in (iv) is a consequence of the fact that category $D^b_{\mathrm{coh}}(\widetilde{X})$ is a Calabi-Yau category. As a result, the algebra $A_{\mathscr{T}}$ is a Calabi-Yau algebra, cf. [Gi3, §7]. This implies the desired statement, see [Br] or [Gi3], Corollary 3.3.2 and Theorem 7.2.14(iii). ☐

One has the following

Conjecture 8.2.3. *For any symplectic resolution as in Theorem 8.1.1, there exists a tilting generator $\mathscr{T}$ such that the algebra $A_{\mathscr{T}}$ has no nonzero homogeneous components of negative degrees.*

One of the motivations for this conjecture is the following observation by D. Kaledin (unpublished), cf. also [BM],

Proposition 8.2.4. *Let $\mathscr{T}$ be a tilting generator $\mathscr{T}$ such that the algebra $A_{\mathscr{T}}$ is nonnegatively graded. Then, A_0 is a semisimple subalgebra of $A_{\mathscr{T}}$, and $A_{\mathscr{T}}$ is a* Koszul *algebra, cf.* [BGS].

Proof. Let $A = \oplus_{i\geq 0}\, A_i$ be any nonnegatively graded algebra $A = \oplus_{i\geq 0}\, A_i$ such that each homogeneous component A_i is finite dimensional and A_0 is a semisimple algebra. Then, any A_0-module may be viewed, via the augmentation $A \twoheadrightarrow A_0$, as an A-module. Using the *minimal* resolution of A_0 by graded projective A-modules shows that the group $\mathrm{Ext}^i_A(A_0, A_0)$ has no nonzero homogeneous components in degrees $j > i$, cf. [BGS]. The Koszul property for A amounts to the condition that, for any $i \geq 0$, the grading on the group $\mathrm{Ext}^i_A(A_0, A_0)$ induced by the grading on A is concentrated in degree i.

Now let $A = A_{\mathscr{T}}$ where $\mathscr{T}$ is a tilting generator such that $A_i = 0$ for all $i < 0$. Then, the proof of Proposition 8.2.1(ii) shows that A_0, the degree zero component, is automatically a semisimple algebra.

Next, fix $i \geq 0$. Observe that the group $\mathrm{Ext}^i_A(A_0, A_0)$ has the natural structure of a graded $\mathbb{C}[X]$-module. Observe further that the variety $\widetilde{X}$ being symplectic it has a trivial canonical bundle. Therefore, the affine variety X is Gorenstein. It follows that, for any $k > 0$, Grothendieck-Serre duality provides a perfect pairing between homogeneous components of $\mathrm{Ext}^i_A(A_0, A_0)$ of degrees $i+k$ and $i-k$, respectively. The component of degree $i+k$ vanishes by the first paragraph of the proof. Hence, the component of degree $i-k$ vanishes as well. □

9. Algebraic cycles and cohomological purity

9.1. Let G be a linear algebraic group and Y a quasi-projective G-variety Y. Let $K^G(Y)$ denote the Grothendieck group of the category of G-equivariant coherent sheaves on Y. This group has a natural structure of $K^G(pt)$-module, where $K^G(pt)$ is identified with the representation ring of G. Given a commutative ring R and a ring homomorphism $K^G(pt) \to R$, we put $K^G_R(Y) := R \otimes_{K^G(pt)} K^G(Y)$.

We let the group G act on $Y \times Y$ diagonally and write $\Delta(Y) \in K^G(Y \times Y)$ for the class of the structure sheaf of the diagonal $Y \subset Y \times Y$.

Definition. We say that Y *has decomposable diagonal in K^G_R-theory* if the class $1 \otimes \Delta(Y) \in K^G_R(Y \times Y)$ is contained in the R-submodule generated by the classes of the form $[\mathcal{E} \boxtimes \mathcal{F}]$, where $\mathcal{E}$ and $\mathcal{F}$ are G-equivariant algebraic vector bundles on Y.

Below, we will be only interested in the cases where G is either trivial or $G = \mathbb{C}^\times$. In the former case we have $K^G(pt) = \mathbb{Z}$; in this case we will use simplified notation $K(-) := K^G(-)$, resp. $K_R(-) := K^G_R(-)$. In the latter case, one has $K^{\mathbb{C}^\times}(pt) = \mathbb{Z}[u, u^{-1}]$.

The usefulness of the notion of decomposable diagonal for our purposes is due to the following well known result.

Lemma 9.1.1. *Let Y be a smooth projective variety with decomposable diagonal in $K_{\mathbb{C}}$-theory. Then, the group $H_\bullet(Y)$ is spanned by the fundamental classes of algebraic cycles.*

Proof. We may assume without loss of generality that Y is connected of dimension d. Let $[Y_{\text{diag}}] \in H_{2d}(Y \times Y)$ be the fundamental of the diagonal in $Y \times Y$.

Recall that one has the (homological) Chern character map $ch : K(Y) \to H_\bullet(Y)$, cf. [Fu]. Standard properties of the Chern character imply, cf. [Fu], that the class $ch(\Delta(Y)) \in H_\bullet(Y \times Y)$ has an expansion of the form $ch(\Delta(Y)) = [Y_{\text{diag}}] + c_{d-1} + \ldots + c_0$, where $c_j \in H_{2j}(Y \times Y)$.

Suppose now that in $K_{\mathbb{C}}(Y \times Y)$ one has an equation of the form $\Delta(Y) = \sum_j \nu_j \otimes [\mathcal{E}_j \boxtimes \mathcal{F}_j]$, for some $\nu_j \in \mathbb{C}$ and some vector bundles $\mathcal{E}_j,\ \mathcal{F}_j$ on Y. Then, applying the Chern character map to the above equation and separating individual homological degrees in the resulting formula, we obtain an equation

$$[Y_{\text{diag}}] = \sum\nolimits_j \alpha_j \otimes e_j \otimes f_j, \qquad \alpha_j \in \mathbb{C},\ e_j, f_j \in H_\bullet(Y),\ \ \deg e_j + \deg f_j = 2d. \tag{9.1.2}$$

Furthermore, for each j, the cohomology classes e_j, f_j are some rational combinations of Chern classes; in particular, these classes are $\mathbb{C}$-linear combinations of the classes of algebraic cycles.

To complete the proof, we exploit convolution in homology. Specifically, we consider the setting of [Gi1, §2] in the special case where $M_1 = M_2 = Y$ and $M_3 = pt$. We let $Z_{12} := Y \times Y$ and $Z_{23} = Y \times pt$. The class $[Y_{\text{diag}}]$ is the unit element of the convolution algebra $H_\bullet(Y \times Y)$. Therefore, for any $c \in H_\bullet(Y) = H_\bullet(Z_{23})$, we have that $[Y_{\text{diag}}] \star c = c$. On the other hand, it is immediate from definitions that for any $e, f \in H_\bullet(Y)$, one has $(e \otimes f) \star c = \langle f, c\rangle \cdot e$, where $\langle -, -\rangle :\ H_\bullet(Y) \times H_{2d-\bullet}(Y) \to \mathbb{C}$ denotes the Poincaré duality pairing. Thus, from equation (9.1.2), we get

$$c = [Y_{\text{diag}}] \star c = (\sum\nolimits_j \alpha_j \otimes e_j \otimes f_j) \star c = \sum\nolimits_j \langle f_j, c\rangle \cdot \alpha_j \otimes e_j.$$

The sum on the right is a $\mathbb{C}$-linear combination of the classes e_j, hence it is a $\mathbb{C}$-linear combination of the classes of algebraic cycles. Thus we have shown that any homology class $c \in H_\bullet(Y)$ is a $\mathbb{C}$-linear combination of the classes of algebraic cycles. □

Lemma 9.1.3. *Let Y be a smooth quasi-projective $\mathbb{C}^\times$-variety. If Y has decomposable diagonal in $K^{\mathbb{C}^\times}$-theory then $Y^{\mathbb{C}^\times}$, the fixed point subvariety, has decomposable diagonal in $K_{\mathbb{C}}$-theory.*

Proof. The result is implicitly contained [CG, Theorem 5.11.10], since the algebra homorphism r_a in *loc cit* sends the unit element to the unit element.

For the benefit of the reader, we spell out the argument in a more explicit way as follows. Put $Y' := Y^{\mathbb{C}^\times}$ and let $i : Y' \hookrightarrow Y$ denote the imbedding. We consider the following diagrams:

$$\begin{tikzcd} Y' \arrow[r, hook, "\Delta'"] \arrow[d, hook, "i"] & Y' \times Y' \arrow[d, hook, "i\times i"] \\ Y \arrow[r, hook, "\Delta"] & Y \times Y \end{tikzcd} \qquad \begin{tikzcd} K_{\mathbb{C}}(Y') \arrow[r, "\Delta'_*"] & K_{\mathbb{C}}(Y' \times Y') \\ K^{\mathbb{C}^\times}(Y) \arrow[u, "r"] \arrow[r, "\Delta_*"] & K^{\mathbb{C}^\times}(Y \times Y) \arrow[u, "r\times r"'] \end{tikzcd} \tag{9.1.4}$$

Here, the maps Δ and Δ' are the diagonal imbeddings, Δ_* and Δ'_* stand for the corresponding push-forward morphisms in K-theory. Finally, r is the map considered in [CG, §5.11], where it has been denoted by res_a and where a stands for an element of the group $\mathbb{C}^\times$ such that $a \neq 1$. The map r was defined in *loc cit* as a composition $\lambda^{-1} \circ ev \circ i^* :\ K^{\mathbb{C}^\times}(Y) \to K^{\mathbb{C}^\times}(Y') \to K_{\mathbb{C}}(Y') \to K_{\mathbb{C}}(Y')$. The first map i^* here is a restriction morphism. The second map $ev :\ K^{\mathbb{C}^\times}(Y') = \mathbb{Z}[u, u^{-1}] \otimes_{\mathbb{Z}} K(Y') \to K_{\mathbb{C}}(Y')$ is induced by the 'evaluation map' $\mathbb{Z}[u, u^{-1}] \to \mathbb{C},\ f \mapsto f(a)$. The third map is given by multiplication by λ^{-1}, an inverse of an equivariant Euler class $\lambda \in K_{\mathbb{C}}(Y')$, which is known to be an invertible element, [CG, Proposition 5.10.3].

The left square in (9.1.4) is clearly commutative. The square on the right commutes also, by [CG, Theorem 5.11.7]. Hence, using that $i^*[\mathcal{O}_Y] = [\mathcal{O}_{Y'}]$, we find

$$(r \times r)(\Delta(Y)) = (r \times r)\Delta_*[\mathcal{O}_Y] = \Delta'_* r([\mathcal{O}_Y]) = \Delta'_*(\lambda^{-1} \cdot [\mathcal{O}_{Y'}]) = (\lambda^{-1} \boxtimes 1) \cdot \Delta'_*[\mathcal{O}_{Y'}]. \tag{9.1.5}$$

Now, since Y has decomposable diagonal in $K^{\mathbb{C}^\times}$-theory, there exist $\mathbb{C}^\times$-equivariant vector bundles $\mathcal{E}_j,\ \mathcal{F}_j$, on Y, such that, in $K^{\mathbb{C}^\times}(Y \times Y)$, one has an equation $\Delta(Y) = \sum_j\ p_j \cdot [\mathcal{E}_j \boxtimes \mathcal{F}_j]$, where $p_j \in \mathbb{Z}[u, u^{-1}]$. Applying the map $r \times r$ to this equation and using (9.1.5) yields

$$(\lambda^{-1} \boxtimes 1) \cdot \Delta'_*[\mathcal{O}_{Y'}] = (r \times r)(\Delta(Y)) = \sum\nolimits_j\ p_j(a) \cdot \big(\lambda^{-1} \cdot ev(i^*[\mathcal{E}_j])\big)\ \boxtimes\ \big(\lambda^{-1} \cdot ev(i^*[\mathcal{F}_j])\big),$$

where $p_j(a)$ are complex numbers. Thus, for the class $\Delta(Y^{\mathbb{C}^\times}) \in K_{\mathbb{C}}(Y^{\mathbb{C}^\times} \times Y^{\mathbb{C}^\times})$, we obtain the following formula

$$\Delta(Y^{\mathbb{C}^\times}) = \Delta'_*[\mathcal{O}_{Y'}] = \sum\nolimits_j\ p_j(a) \cdot ev(i^*[\mathcal{E}_j])\ \boxtimes\ \big(\lambda^{-1} \cdot ev(i^*[\mathcal{F}_j])\big).$$

It is manifest from the formula that the variety $Y^{\mathbb{C}^\times}$ has decomposable diagonal in $K_{\mathbb{C}}$-theory. $\square$

9.2. We can now prove the main result of this subsection.

Theorem 9.2.1. *Let $\pi : \widetilde{X} \to X$ be a symplectic resolution satisfying our standing assumptions. Then,*

(i) *The $\mathbb{C}^\times$-variety $\widetilde{X}$ has decomposable diagonal in $K^{\mathbb{C}^\times}$-theory.*

(ii) *The groups $H_\bullet(\pi^{-1}(0))$ and $H^{BM}_\bullet(\widetilde{X})$, the homology of the central fiber and the Borel-Moore homology of $\widetilde{X}$, respectively, are generated by the fundamental classes of algebraic cycles.*

(iii) *The cohomology groups $H^i(X)$ vanish whenever i is odd or $i > \dim_{\mathbb{C}} X$. Furthermore, for any $i \geq 0$, the Hodge structure on the cohomology $H^{2i}(X, \mathbb{C})$ is a (pure) Tate structure of type (i, i).*

Remark 9.2.2. On can show that, more generally, the statements of part (iii) hold for the cohomology of any fiber $\pi^{-1}(x)$ of the map π.

Proof. Thanks to Theorem 8.1.1, one can find a $\mathbb{C}^\times$-equivariant vector bundle $\mathscr{T}$, on $\widetilde{X}$, which is a tilting generator. Let $\mathscr{T}^*$ be the dual vector bundle and let $A := A_{\mathscr{T}} \cong \mathscr{T} \otimes \mathscr{T}^*$.

First of all, from Theorem 8.1.1 one deduces that the functor $\mathrm{Hom}(\mathscr{T} \boxtimes \mathscr{T}^*, -)$ provides an equivalence $D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(\widetilde{X} \times \widetilde{X}) \xrightarrow{\sim} D^b(A\text{-}\mathsf{grbimod})$, cf. Remark 8.1.2. Convolution with the object $\mathcal{O}_{\widetilde{X}_{\mathrm{diag}}}$, the structure sheaf of the diagonal, acts on the category $D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(\widetilde{X})$ as the identity functor. Similarly, tensoring with $A_{\mathrm{diag}} \in A\text{-}\mathsf{grbimod}$, the diagonal A-bimodule, acts on the category $D^b(A\text{-}\mathsf{grmod})$ as the identity functor. It follows that the equivalence above sends $\mathcal{O}_{\widetilde{X}_{\mathrm{diag}}}$ to A_{diag}.

We choose a graded resolution $\ldots \to P_1 \to P_0 \twoheadrightarrow A_{\mathrm{diag}}$ of the diagonal bimodule by free $A \otimes A^{op}$-modules of finite rank. It follows from Proposition 8.1.1(i) that the algebra $A \otimes A^{op}$ has finite global dimension. This implies by a standard argument involving long exact sequences of Ext-groups that there exists an integer $n \gg 0$ such that, writing $K := \mathrm{Ker}[P_n \to P_{n-1}]$, in the category of all A-bimodules one has $\mathrm{Ext}^1(K, M) = 0$, for any A-bimodule M. Hence, K is projective as an ungraded $A \otimes A^{op}$-module. Therefore, thanks to Proposition 8.1.1(iii), one has an isomorphism $K = \oplus_{j=1}^m K'_j \boxtimes K''_j$, where each K'_j, resp. K''_j, is a direct summand of A viewed as a rank one free A-module, resp. A^{op} viewed as a rank one free A^{op}-module. Thus, we obtain a graded resolution

$$0 \to K \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \twoheadrightarrow A_{\mathrm{diag}}$$

such that each of the objects $P_0, P_1, \ldots, P_n, K$, is a finite direct sum of objects of the form $P' \boxtimes P''$, where P' and P'' are direct summands of a rank one free module. It follows that $[A_\Delta]$, the class of A_Δ in the Grothendieck group $K(A\text{-}\mathsf{grbimod})$, is equal to a linear combination of classes of the form $[P'] \boxtimes [P'']$ where P' and P'' are direct summands of a rank one free module.

Now the equivalence of Theorem 8.1.1 sends $A \in A\text{-}\mathsf{grmod}$ to $\mathscr{T} \in D^{b,\mathbb{C}^\times}_{\mathrm{coh}}(\widetilde{X})$. Since $\mathscr{T}$ is a vector bundle, the equivalence sends any direct summand of A to a $\mathbb{C}^\times$-equivariant vector bundle on $\widetilde{X}$. Further, our equivalences of triangulated categories induce isomorphisms of the corresponding Grothendieck groups. We have shown that the class $[A_\Delta] \in K(A\text{-}\mathsf{grbimod})$ goes, under the isomorphism $K(A\text{-}\mathsf{grbimod}) \cong K^{\mathbb{C}^\times}(\widetilde{X} \times \widetilde{X})$, to the class $\Delta(\widetilde{X}) \in K^{\mathbb{C}^\times}(\widetilde{X} \times \widetilde{X})$. Combining all the above proves part (i) of the theorem.

Observe next that, the resolution $\pi : \widetilde{X} \to X$ being $\mathbb{C}^\times$-equivariant, we have $\pi(\widetilde{X}^{\mathbb{C}^\times}) \subset X^{\mathbb{C}^\times} = \{o\}$. Hence, one has an inclusion $\widetilde{X}^{\mathbb{C}^\times} \subset \pi^{-1}(o)$, which shows that $\widetilde{X}^{\mathbb{C}^\times}$ is a projective variety. Applying Lemma 9.1.3, we deduce that this variety has decomposable diagonal in $K_{\mathbb{C}}$-theory. Thus, the group $H_\bullet(\widetilde{X}^{\mathbb{C}^\times})$ is spanned by algebraic cycles, thanks to Lemma 9.1.1.

To complete the proof of part (ii), one uses a standard argument based on the Bialynicki-Birula decomposition. In more detail, we first choose a smooth $\mathbb{C}^\times$-equivariant completion Y of $\widetilde{X}$, as in the proof of Proposition 4.6.1. Then, the argument used in the proof of that Proposition shows that $\widetilde{X}$ is a union of some of the attracting pieces of the Bialynicki-Birula decomposition of the variety Y. A similar argument shows that $\pi^{-1}(0)$ is a union of some of the *expanding* pieces of the Bialynicki-Birula decomposition for Y. Next, one proves by a simple argument based on long exact sequences, cf. [CG, Lemma 5.9.20], that the fact that the homology of $\widetilde{X}^{\mathbb{C}^\times}$ is spanned by the algebraic cycles implies a similar result for the varieties $\widetilde{X}$ and $\pi^{-1}(0)$. Part (ii) follows.

Now, it is a direct consequence of (ii) that the cohomology groups $H^i(X)$ vanish for i odd and the mixed Hodge structure on $H^{2i}(X)$ is a (pure) Tate structure of weight i. Finally, we know that the varieties $\widetilde{X}$ and $\pi^{-1}(0)$ are homotopy equivalent. It follows that $H^i(\widetilde{X}) = H^i(\pi^{-1}(0)) = 0$. for all $i > 2\dim \pi^{-1}(0)$. On the other hand, Theorem 4.2.1(2) yields an inequality $\dim \pi^{-1}(0) \leq \frac{1}{2} \dim \widetilde{X}$. Part (iii) follows. This completes the proof of the theorem. $\square$

Remark 9.2.3. The odd homology vanishing and generation of homology by algebraic cycles for the fibers of the Springer resolution, equivalently, for the e-fixed point varieties $\mathcal{B}_e \subset \mathcal{B}$, was standing as an open problem for quite a long time. This problem has been finally solved in [DCLP]. The argument in [DCLP] was quite technical, in particular, it involved a case-by-case analysis.

The fact that the variety $\widetilde{\mathcal{S}}_e$ has a decomposable diagonal in K-theory, for any nilpotent element in an arbitrary semisimple Lie algebra $\mathfrak{g}$, has not been known until the result of Kaledin [K5].

The odd cohomology vanishing for the fibers of the map $\mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathcal{M}_0(\mathbf{v},\mathbf{w})$ was proved in [Nak].

9.3. **Applications to counting over finite fields.** There is a counterpart of the above result over finite fields. Specifically, assume that the variety X is obtained by base change from a scheme X_S over S, where S is a Zariski open subset of $\operatorname{Spec}\mathbb{Z}$. For any prime $p \in S$, we get by reduction modulo p a scheme X_p over $\mathbb{F}_p = \mathbb{Z}/(p)$, the residue field. the reduction of X modulo p. Thus, for each $n = 1, 2, \ldots$, one has a finite set $X(\mathbb{F}_{p^n})$ of $\mathbb{F}_{p^n}$-rational points of the scheme X_p.

Write $H^i_{et}(X_p, \bar{\mathbb{Q}}_\ell)$ for ℓ-adic étale cohomology of the scheme X_p, where $\ell \in \mathbb{Z}$ is a prime, $\ell \neq p$. The étale cohomology groups come equipped with an action of the Frobenius endomorphism. Using the comparison theorem for étale cohomology and the Grothendieck-Lefschetz fixed point formula, from Theorem 9.2.1(ii) one derives the following result.

Theorem 9.3.1. *For all but finitely many primes $p \in S$, the following holds:*

(i) *The scheme X_p is smooth.*

(ii) *For any $i \geq 0$, the Frobenius endomorphism acts on $H^{2i}_{et}(X_p, \bar{\mathbb{Q}}_\ell)$ as multiplication by p^i; furthermore, one has $H^i_{et}(X_p, \bar{\mathbb{Q}}_\ell) = 0$ whenever i is odd or $i > d := \dim X$.*

(ii) *The number of elements of the set $X(\mathbb{F}_{p^n})$ is given by the formula*

$$|X(\mathbb{F}_{p^n})| \;=\; \sum_{i=0}^{\dim X} \dim H^{2(d-i)}_{et}(X_p, \bar{\mathbb{Q}}_\ell) \,\cdot\, p^{i\cdot n}, \qquad \forall n \geq 1. \qquad \square$$

Remark 9.3.2. In the special case of quiver varieties $\mathcal{M}_\theta(\mathbf{v}, 0)$ such that the dimension vector $\mathbf{v} = (v_i)_{i\in I}$ is indivisible (i.e. such that $\gcd\big((v_i)_{i\in I}\big) = 1$) and, moreover, the stabilty condition θ is sufficiently general, the above theorem was first obtained by Crawley-Boevey and Van den Bergh [CBV]. A closely related result was later obtained by Hausel in his proof of a conjecture by V. Kac, see [Ha] and references therein.

10. Appendix 1: On rational singularities

10.1. **Setup.** Let X be a normal variety. We consider a diagram

$$\widetilde{X} \xrightarrow{\ \pi\ } X \xleftarrow{\ j\ } U$$

where π is a resolution of singularities and j is a Zariski open imbedding of a set U contained in the smooth locus of X and such that $X \smallsetminus U$ has codimension ≥ 2 in X.

One can take U to be the regular locus of X, for instance.

Theorem 10.1.1. *Let Ω be a regular nowhere vanishing volume form on U and assume that the form $\pi^*\Omega$, on $\pi^{-1}(U)$, can be extended to a regular form $\widetilde{\Omega}$ on $\widetilde{X}$. Then, we have:*

(i) *The variety X is Cohen-Macaulay;*

(ii) *The dualizing sheaf of X is the structure sheaf $\mathcal{O}_X$;*

(iii) *One has $R^k\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for all $k \neq 0$, i.e., X has rational singularities.*

Remark 10.1.2. The form $\widetilde{\Omega}$, in the theorem, is allowed to have zeros on $\widetilde{X} \smallsetminus \pi^{-1}(U)$.

The above theorem, due to Flenner [Fl, Satz 1.3], may be deduced from a more general result by R. Elkik [El]. Several proofs of various generalizations of the theorem have appeared in the literature, cf. eg. [KKM], [Kov]. Below we give a streamlined selfcontained proof of Theorem 10.1.1 following the strategy of M. Kovács [Kov].

10.2. **Proof of Theorem 10.1.1.** *Step 1.* Let $\mathcal{K}_Y$ denote the canonical sheaf of a smooth variety Y.

The assignment $1 \mapsto \Omega$, resp. $1 \mapsto \widetilde{\Omega}$, gives a sheaf isomorphism $v:\ \mathcal{O}_U \xrightarrow{\sim} \mathcal{K}_U$, resp. a sheaf morphism $\widetilde{v}:\ \mathcal{O}_{\widetilde{X}} \to \mathcal{K}_{\widetilde{X}}$.

A key role in the argument is played by the following diagram

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rrr, dotted, "\mathrm{Id}"] \arrow[d, "\alpha"] & & & \mathcal{O}_X \arrow[r, "\delta", "\cong"'] & j_*\mathcal{O}_U \arrow[d, "\cong"', "j_*(v)"] \\
\pi_*\mathcal{O}_{\widetilde{X}} \arrow[r, "\pi_*(\widetilde{v})"] & \pi_*\mathcal{K}_{\widetilde{X}} \arrow[urr, dotted, "\beta"] \arrow[rr, "\mathrm{adj}"] & & j_*j^*\pi_*\mathcal{K}_{\widetilde{X}} \arrow[r, equal, "\gamma"] & j_*\mathcal{K}_U
\end{tikzcd} \tag{10.2.1}$$

In this diagram, the maps α and adj are canonical adjunctions, the isomorphism γ follows from $j^*\pi_*\mathcal{K}_{\widetilde{X}} = \mathcal{K}_U$, and the isomorphism δ holds since $X \smallsetminus U$ has codimension ≥ 2 and X is normal. The dotted arrow β is by definition given by a composition $\beta := \delta^{-1} \circ j_*(v)^{-1} \circ \gamma$.

It is clear that, writing $1 \in \mathcal{O}_X$ for the unit section, one has

$$1 \xmapsto{\pi_*(\widetilde{v})\circ\alpha} \pi_*\widetilde{\Omega} \xmapsto{\gamma\circ\mathrm{adj}} j_*\Omega \xleftarrow{j_*(v)\circ\delta} 1$$

It follows that the composition $\beta \circ \pi_*(\widetilde{v}) \circ \alpha:\ \mathcal{O}_X \to \mathcal{O}_X$, along the perimeter of diagram (10.2.1), equals the identity map $\mathcal{O}_X \to \mathcal{O}_X$. We deduce, in particular, that the morphism $\beta:\ \pi_*\mathcal{K}_{\widetilde{X}} \to \mathcal{O}_X$ is surjective. This morphism is also injective since its restriction to U is an isomorphism and the sheaf $\pi_*\mathcal{K}_{\widetilde{X}}$ is clearly torsion free. Thus, β is an isomorphism.

Step 2. The morphisms α and $\pi_*(\widetilde{v})$ have natural derived analogues $R\alpha$ and $R\pi_*(\widetilde{v})$, respectively. Thus, in the derived category, there is a chain of morphisms

$$\mathcal{O}_X \xrightarrow{R\alpha} R\pi_*\mathcal{O}_{\widetilde{X}} \xrightarrow{R\pi_*(\widetilde{v})} R\pi_*\mathcal{K}_{\widetilde{X}} \xrightarrow[\sim]{\mathrm{GR}} \pi_*\mathcal{K}_{\widetilde{X}} \xrightarrow[\sim]{\beta} \mathcal{O}_X. \tag{10.2.2}$$

Here, 'GR' is the canonical quasi-isomorphism provided by the Grauert-Riemenschnider theorem and the morphism β is the isomorphism of Step 1.

Let $\phi := \beta \circ \mathrm{GR} \circ R\pi_*(\widetilde{v})\ :\ R\pi_*\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$ be the composite morphism. With this notation, diagram (10.2.2) reads

$$\mathcal{O}_X \xrightarrow{R\alpha} R\pi_*\mathcal{O}_{\widetilde{X}} \xrightarrow{\phi} \mathcal{O}_X. \tag{10.2.3}$$

It is clear that the composition $\phi \circ R\alpha$, in (10.2.3), still sends 1 to 1. Hence this composition is equal to the identity.

Step 3. Let D_X denote the dualizing complex of the scheme X. This is an object of the derived category, and we use an unconventional normalization such that $\mathscr{H}^0(D_X)|_U = \mathcal{K}_U$. Write $\mathbb{D}$ for the Grothendieck duality functor, normalized accordingly. Thus, $D_X = \mathbb{D}(\mathcal{O}_X)$ and $\mathcal{K}_X = \mathbb{D}(\mathcal{O}_{\widetilde{X}})$ since $\widetilde{X}$ is smooth. Grothendieck's duality commutes with proper push-forward, so we have

$$\mathbb{D}(R\pi_*\mathcal{O}_{\widetilde{X}}) = R\pi_*(\mathbb{D}(\mathcal{O}_{\widetilde{X}})) = R\pi_*\mathcal{K}_{\widetilde{X}}. \tag{10.2.4}$$

Now, we apply Grothendieck's duality to diagram (10.2.3) and use the composite isomorphism in (10.2.4). This way, one obtains a diagram

$$D_X \xrightarrow{\mathbb{D}(\phi)} R\pi_*\mathcal{K}_{\widetilde{X}} \xrightarrow{\mathbb{D}(R\alpha)} D_X.$$

By Step 2, the composite morphism above is the identity morphism $D_X \to D_X$. On the other hand, we know that $R\pi_*\mathcal{K}_{\widetilde{X}} = \pi_*\mathcal{K}_{\widetilde{X}}$, by the Grauert-Riemenschnider theorem [GR]. We conclude that the identity morphism of the complex D_X factors through a morphism to a complex concentrated in degree zero. This forces $\mathscr{H}^k(D_X) = 0$ for all $k \neq 0$.[2] Thus, X is Cohen-Macaulay and part (i) of the theorem follows.

To prove parts (ii) and (iii), we apply Grothendieck's duality to (10.2.4). Using the composite isomorphism $\beta \circ \mathrm{GR}:\ R\pi_*\mathcal{K}_{\widetilde{X}} \xrightarrow{\sim} \pi_*\mathcal{K}_{\widetilde{X}} \xrightarrow{\sim} \mathcal{O}_X$, see (10.2.2), one obtains $R\pi_*\mathcal{O}_{\widetilde{X}} = \mathbb{D}^2(R\pi_*\mathcal{O}_{\widetilde{X}}) = \mathbb{D}(R\pi_*\mathcal{K}_{\widetilde{X}}) = \mathbb{D}(\mathcal{O}_X) = D_X$. Therefore, for all $k \neq 0$, we deduce $R^k\pi_*\mathcal{O}_{\widetilde{X}} = \mathscr{H}^k(D_X) = 0$. For $k = 0$, one finds $\mathcal{O}_X = \pi_*\mathcal{O}_{\widetilde{X}} = D_X$, where the first isomorphism follows from the Zariski main theorem since X is normal. □

11. Appendix 2: Reminder on GIT and stability

A general theory of quotients by a reductive group action via stability conditions has been developed by D. Mumford, and is called Geometric Invariant Theory, cf. [**?**]. For a 'reader friendly' exposition of the subject we recommend [**?**]; cf. also [**?**] for a more differential-geometric approach.

11.1. Throughout the paper, the ground field is the field $\mathbb{C}$ of complex numbers. We write $\otimes = \otimes_{\mathbb{C}}$ and $\dim = \dim_{\mathbb{C}}$.

Let X be a not necessarily irreducible, *affine* algebraic G-variety, where G is a reductive linear algebraic group. Given a rational character (= algebraic group homomorphism) $\chi : G \to \mathbb{C}^\times$, not of finite order, Mumford defines a scheme $X/\!/_\chi G$ in the following way. Let G act on the cartesian product $X \times \mathbb{C}$ by the formula $g:\ (x,z) \mapsto (gx, \chi(g)^{-1}\cdot z)$ (more generally, the cartesian product $X \times \mathbb{C}$ may be replaced here by the total space of any G-equivariant line bundle on X). The coordinate ring of $X \times \mathbb{C}$ is the algebra $\mathbb{C}[X \times \mathbb{C}] = \mathbb{C}[X] \otimes \mathbb{C}[z]$, of polynomials in a variable z with coefficients in the coordinate ring of X. This algebra has an obvious grading by degree of the polynomial.

Let $A_\chi := \mathbb{C}[X \times \mathbb{C}]^G$ be the subalgebra G-invariants. Clearly, this is a graded subalgebra which is, moreover, a finitely generated algebra by Hilbert's theorem on finite generation of algebras of invariants, cf. [**?**, ch. II, §3.1]. Explicitly, a polynomial $f(z) = \sum_{n=0}^N f_n \cdot z^n \in \mathbb{C}[X] \otimes \mathbb{C}[z]$ is G-invariant if and only if, for each $n = 0, \ldots, N$, the function f_n is a χ^n-*semi-invariant*, i.e. if and only if one has

$$f_n(g^{-1}(x)) = \chi(g)^n \cdot f_n(x), \quad \forall g \in G,\ x \in X.$$

Write $\chi^n:\ g \mapsto \chi(g)^n$ for the n-th power of the character χ and let $\mathbb{C}[X]^{\chi^n} \subset \mathbb{C}[X]$ be the vector space of χ^n-semi-invariant functions. It is clear that we have

$$A_\chi := \mathbb{C}[X \times \mathbb{C}]^G = \bigoplus\nolimits_{n \geq 0} \mathbb{C}[X]^{\chi^n},$$

and the direct sum decomposition on the right corresponds to the grading on the algebra A_χ.

Let $X/\!/_\chi G := \mathsf{Proj}\, A_\chi$ be the projective spectrum of the graded algebra A_χ. This is a quasi-projective scheme, called a *GIT quotient* of X by the G-action. The scheme $X/\!/_\chi G$ is reduced,

[2]This trick, apparently due to Kovács, has been used in [Kov] to give elegant new proofs of other important known results. In particular, Kovács reproves that any canonical singularity is rational. He also proves that a categorical quotient of a normal variety with rational singularities by a reductive group action is again a variety with rational singularities, generalizing a well-known earlier result of Hochster and Roberts.

resp. irreducible, normal, whenever so is X (since A_χ has no nilpotents, resp. no zero divisors, is integrally closed, provided this holds for $\mathbb{C}[X]$).

Put $A_\chi^{>0} := \bigoplus_{n>0} \mathbb{C}[X]^{\chi^n}$. Let $\mathscr{I}$ be the set of homogeneous ideals $I \subset A_\chi$ such that one has $I \neq A_\chi$ and $A_\chi^{>0} \not\subset I$. An ideal $I \in \mathscr{I}$ is said to be a 'maximal homogeneous ideal' if it is not properly contained in any other ideal $I' \in \mathscr{I}$. Geometric points of the scheme $X/\!/_\chi G$ correspond to the maximal homogeneous ideals.

In general, for $n = 0$, we have $\mathbb{C}[X]^{\chi^n} = \mathbb{C}[X]^G$, is the algebra of G-invariants. Thus, we have a canonical algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_\chi$ as the degree zero subalgebra. Put another way, the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X \times \mathbb{C}]^G = A_\chi$ is induced by the first projection $X \times \mathbb{C} \to X$.

Standard results of algebraic geometry imply that the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_\chi$ induces a *projective* morphism of schemes $\pi : \ \mathsf{Proj}\, A_\chi \to \operatorname{Spec} \mathbb{C}[X]^G = X/\!/G$.

Remark 11.1.1. In the special case where $G = \mathbb{C}^\times$ and $A = \mathbb{C}[u_0, u_1, \ldots, u_m]$, is a polynomial algebra, we have $\mathsf{Proj}\, A = \mathbb{P}^m = (\mathbb{C}^{m+1} \smallsetminus \{0\})/\mathbb{C}^\times$.

More generally, let G be a reductive group and $\chi : G \twoheadrightarrow \mathbb{C}^\times$ a surjective character. Thus, $K := \operatorname{Ker} \chi$ is a reductive normal subgroup of G and χ induces an isomorphism $G/K \xrightarrow{\sim} \mathbb{C}^\times$.

Now, let X be an affine G-variety such that $\mathbb{C}[X]^{\chi^n} = 0$ for all $n < 0$. Let $X/\!/K = \operatorname{Spec}(\mathbb{C}[X]^K)$, a categorical quotient by K. The residual action of the group G/K on algebra $\mathbb{C}[X]^K$ gives a $\mathbb{Z}$-grading $\mathbb{C}[X]^K = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}[X]^K_n$. Equivalently, there is a natural residual action of the group $\mathbb{C}^\times$ on $X/\!/K$, the categorical quotient of X by the K-action. By definition, we have $\mathbb{C}[X]^K_n = \mathbb{C}[X]^{\chi^n}$. Our assumptions imply that the grading on $\mathbb{C}[X]^K$ is nonnegative and we have $A_\chi = \mathbb{C}[X]^K$.

Thus, we obtain $X/\!/_\chi G = \mathsf{Proj}\, A_\chi \cong \mathsf{Proj}(\mathbb{C}[X]^K)$. Furthermore, the $\mathbb{C}^\times$-action on $X/\!/K$ is a contraction to $Y := (X/\!/K)^{\mathbb{C}^\times}$, the $\mathbb{C}^\times$-fixed point locus, and geometric points of the scheme $\mathsf{Proj}(\mathbb{C}[X]^K)$ correspond to the $\mathbb{C}^\times$-orbits in $(X/\!/K) \smallsetminus Y$. $\Diamond$

Remark 11.1.2. For any character $G \to \mathbb{C}^\times$ and any positive integer $m > 0$, one may view the algebra A_{χ^m} as a graded subalgebra in A_χ via the natural imbedding $A_{\chi^m} = \bigoplus_{\{n\geq 0,\ m|n\}} \mathbb{C}[X]^{\chi^n} \hookrightarrow A_\chi = \bigoplus_{n\geq 0} \mathbb{C}[X]^{\chi^n}$, called the *Veronese imbedding*. One can show that the Veronese imbedding induces an isomorphism $X/\!/_\chi G \xrightarrow{\sim} X/\!/_{\chi^m} G$, of algebraic varieties. $\Diamond$

Given a nonzero homogeneous semi-invariant $f \in A_\chi$ we put $X_f := \{x \in X \mid f(x) \neq 0\}$. To get a better understanding of the GIT quotient $X/\!/_\chi G$, one introduces the following definition, **[?]**.

Definition. (i) A point $x \in X$ is called χ-*semistable* if there exists $n \geq 1$ and a χ^n-semi-invariant $f \in \mathbb{C}[X]^{\chi^n}$ such that $x \in X_f$.

(ii) A point $x \in X$ is called χ-*stable* if there exists n and f as in (i) such that $x \in X_f$ and, in addition, all points of X_f have finite stabilizers.

Write X_χ^{ss}, resp. X_χ^s, for the set of semistable, resp. stable, points. Thus, we have $X_\chi^s \subset X_\chi^{ss} \subset X$.

(iii) Semistable points x and x' are called *S-equivalent* if and only if the orbit closures $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ meet in X_χ^{ss}.

Each of the sets X_χ^{ss} and X_χ^s is a union of sets of the form X_f, hence it is a G-stable Zariski open subset of X. Furthermore, there is a well defined morphism $\varpi : \ X_\chi^{ss} \to X/\!/_\chi G$, of algebraic varieties, obtained by gluing together the usual categorical quotient maps $\varpi_f : X_f \to X_f/\!/G$, for various semi-invariants f. The resulting map ϖ is constant on G-orbits and the image of a G-orbit $\mathcal{O} \subset X_\chi^{ss}$ is a point corresponding to the maximal homogeneous ideal $\mathfrak{I}_\mathcal{O} \subset A_\chi$ formed by the functions $f \in A_\chi$ such that $f(\mathcal{O}) = 0$.

Let Z be the set of points $x \in X_\chi^{ss}$ such that the stabilizer of x is a group of dimension > 0. Let $U := (X/\!/_\chi G) \smallsetminus \varpi(Z)$.

One of the basic results of GIT reads

Theorem 11.1.3. (i) *The morphism ϖ induces a natural bijection between the set of S-equivalence classes of G-orbits in X^{ss}_χ and the set of geometric points of the scheme $X/\!/_\chi G$.*

(ii) *The set U is Zariski open (not necessarily dense) in $X/\!/_\chi G$ and we have $X^s_\chi = \varpi^{-1}(U)$. Moreover, each fiber of the map $\varpi:\ X^s_\chi \to U$ is a single G-orbit in X^{ss}_χ of maximal dimension; thus one has a diagram*

$$\begin{tikzcd} & X^s_\chi = \varpi^{-1}(U) \arrow[r, two heads, "\varpi"] \arrow[d, hook] & U \arrow[d, hook] \\ X & X^{ss}_\chi \arrow[l, hook', "j"'] \arrow[r, two heads, "\varpi"] & X/\!/_\chi G \end{tikzcd} \tag{11.1.4}$$

Sketch of Proof. We let G act on $\widetilde{X} := X \times \mathbb{C}$ as at the beginning of the section. We consider the adjoint quotient map $p:\ \widetilde{X} \to \widetilde{X}/\!/G$ and, for any $x \in X$, consider the point $(x,1) \in \widetilde{X}$. Then, Definition 11.1 says that x is χ-semistable if and only if one has $p(x,1) \notin p(X \times \{0\})$.

Using this interpretation, all the statements of the theorem become easy consequences of the fact that the assignment $Y \mapsto p^{-1}(Y)$ yields a bijection between closed subsets of $\widetilde{X}/\!/G$ and G-stable closed subsets of $\widetilde{X}$. In particular, we deduce from the above that x is semistable if and only if the closure of the G-orbit of the element $(x,1) \in \widetilde{X}$ does not meet $X \times \{0\}$. It follows that elements $x, x' \in X^{ss}_\chi$ are S-equivalent if and only if the closures of the G-orbits of $(x,1)$ and of $(x',1)$ meet in $\widetilde{X} \smallsetminus (X \times \{0\})$. This implies (i).

To prove (ii), let $f \in \mathbb{C}[X]^{\chi^n}$, $n > 0$, be a semi-invariant such that all points in X_f have finite stabilizers. Let $\widetilde{f} = f \cdot z^n \in \mathbb{C}[X] \otimes \mathbb{C}[z]$ be the corresponding G-invariant function on $X \times \mathbb{C}$. It follows that, for any $c \neq 0$ all points of the level set $\widetilde{f}^{-1}(c)$ have finite stabilizers. Thus, $\widetilde{f}^{-1}(c)$ is a G-stable *closed* subset of $\widetilde{X}$ such that any G-orbit contained in $\widetilde{f}^{-1}(c)$ has maximal dimension, $\dim G$. Hence, the closure of such an orbit in $\widetilde{f}^{-1}(c)$ must be equal to the orbit itself, i.e. all orbits in $\widetilde{f}^{-1}(c)$ are closed. This implies that, for any fiber W of the map $p:\ \widetilde{X} \to \widetilde{X}/\!/G$, the set $W \cap \widetilde{f}^{-1}(c)$ is either a single G-orbits or empty.

Now, let $U_f := \varpi(X_f)$. We deduce that each fiber of the map $\varpi:\ \varpi^{-1}(U_f) \to U_f$ is a single G-orbit. Thus, we have $\varpi^{-1}(U_f) = X_f$ and therefore $X_f \subset \varpi^{-1}(U)$, since any point of X_f has a finite stabilizer. We conclude that $X^s_\chi \subset \varpi^{-1}(U)$. Conversely, any point of $\varpi^{-1}(U)$ has a finite stabilizer, by definition. This completes the proof. ☐

Examples 11.1.5. (i) For the trivial character $\chi = 1$, we have $A_\chi = \mathbb{C}[X]^G \otimes \mathbb{C}[z]$. The regular function $z \in A_\chi$ is a homogeneous degree one regular function that does not vanish on X. Therefore, we have $X = X_z$ and any point $x \in X$ is χ-semistable. Such a point is χ-stable if and only if the G-orbit of x is a closed orbit in X of dimension $\dim G$. Furthermore, one has

$$X/\!/_\chi G \;=\; \mathsf{Proj}\, A_\chi \;=\; \mathsf{Proj}\,\big(\mathbb{C}[X]^G \otimes \mathbb{C}[z]\big) \;=\; \operatorname{Spec} \mathbb{C}[X]^G \;=\; X/\!/G, \ \text{ for } \ \chi = 1.$$

In this case, the canonical map π becomes an isomorphism $X/\!/_\chi G \xrightarrow{\sim} X/\!/G$.

(ii) Let G be a connected semisimple group with Lie algebra $\mathfrak{g}$. We put $\widetilde{G} := G \times \mathbb{C}^\times$ and let $\chi:\ \widetilde{G} \to \mathbb{C}^\times$ be the character $\chi(g,z) := z$. We let G act on $\mathfrak{g}$ via the adjoint action and let the group $\mathbb{C}^\times$ act on $\mathfrak{g}$ by dilations. This makes $\mathfrak{g}$ a $\widetilde{G}$-variety. It is clear that $\mathbb{C}[\mathfrak{g}]^{\chi^n}$ is the space of homogeneneous $\operatorname{Ad} G$-invariant polynomials on $\mathfrak{g}$ of degree n. Thus, we have $A_\chi = \mathbb{C}[\mathfrak{g}]^G$ and also $\operatorname{Ker} \chi = G$. Therefore, $\mathfrak{g}/\!/_\chi \widetilde{G} = \mathsf{Proj}(\mathbb{C}[\mathfrak{g}]^G) = ((\mathfrak{g}/\!/ \operatorname{Ad} G) \smallsetminus \{0\})/\mathbb{C}^\times$.

Further, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let W be the corresponding Weyl group. By a well-known result of Chevalley, the algebra $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to $\mathbb{C}[\mathfrak{h}]^W$ and, moreover, the latter is a free polynomial algebra. We deduce that $\mathfrak{g}//_\chi\widetilde{G}$ is isomorphic to $\mathbb{P}(\mathfrak{h}/W) := (\mathfrak{h}/W \smallsetminus \{0\})/\mathbb{C}^\times$, a *weighted* projective space.

Finally, we observe that an element $x \in \mathfrak{g}$ is semistable if and only if it is not nilpotent and it is stable if and only if it is a regular semisimple element of $\mathfrak{g}$.

We will frequently use the following result which is, essentially, a consequence of definitions.

Corollary 11.1.6. (i) *Let X be a smooth G-variety such that the isotropy group of any point of X is connected. Then the set $\varpi(X^s_\chi)$ is contained in the smooth locus of the scheme $X//_\chi G$.*

(ii) *Assume, in addition, that X is affine and that the G-action on X^{ss}_χ is free. Then any semistable point is stable, the scheme $X//_\chi G$ is smooth. Furthermore, the morphism $\varpi : X^{ss}_\chi \to X//_\chi G$ is a principal G-bundle (in ètale topology).* $\square$

In the situation of part (ii) of the Corollary, one often calls the map ϖ, or the variety $X//_\chi G$, a *universal geometric quotient*.

11.2. Categories of coherent sheaves on the G-variety X and its GIT quotients are related via natural functors induced by the morphisms in diagram (11.1.4). In general, let $\mathrm{Coh}(Y)$, resp. $\mathrm{Coh}^G(Y)$, denote the abelian category of coherent sheaves, resp. G-equivariant coherent sheaves, on a scheme Y. Then, pull-back and push-forward via the map ϖ give functors $\mathrm{Coh}(X/\!\!/_\chi G) \to \mathrm{Coh}^G(X^{ss}_\chi)$ and $\mathrm{Coh}^G(X^{ss}_\chi) \to \mathrm{Coh}(X/\!\!/_\chi G)$, where the second functor sends a sheaf $\mathscr{F}$ to $(\varpi_*\mathscr{F})^G$.

Further, given a finitely generated graded A_χ-module M and a semi-invariant f on X, one obtains by localization a G-equivariant coherent sheaf on the open set $X_f \subset X$. Taking G-invariants gives a coherent sheaf on $X_f//G$. It is a formal consequence of the Proj-construction that the resulting sheaves on open sets $X_f//G$ glue together, for various semi-invariants f, to produce a well defined coherent sheaf $\mathbb{F}(M)$ on the scheme $\mathsf{Proj}\, A_\chi$.

Let A_χ-grmod be the abelian category finitely generated $\mathbb{Z}$-graded left A_χ-modules and A_χ-tails the full subcategory of A_χ-grmod whose objects are the A_χ-modules which have only finitely many nonzero homogeneous components. It is immediate from definitions that A_χ-tails is a *Serre subcategory* (i.e. and abelian subcategory stable under taking extensions) of A_χ-grmod.

Remark 11.2.1. We remark that since the grading on the algebra A_χ is nonnegative, *any* finitely generated $\mathbb{Z}$-graded A_χ-module has at most finitely many nonzero homogeneous components of *negative* degrees.

The construction above yields a functor $\mathbb{F} :\ A_\chi\text{-}\mathsf{grmod} \to \mathrm{Coh}(X/\!\!/_\chi G),\ M \mapsto \mathbb{F}(M)$. This functor kills any object of A_χ-tails, by definition. Moreover, one has the following generalization of a classical result of Serre:

Theorem 11.2.2. *The functor $\mathbb{F}$ induces an equivalence*

$$\mathbb{F} :\ A_\chi\text{-}\mathsf{grmod}/A_\chi\text{-}\mathsf{tails} \xrightarrow{\sim} \mathrm{Coh}(X/\!\!/_\chi G).$$

The equivalence of the theorem is related to pull-back and push-forward functors resulting from diagram (11.1.4) as follows

Proposition 11.2.3. (i) *For any $\mathscr{S} \in \mathrm{Coh}^G(X)$, there exists a large enough integer $m(\mathscr{S})$ such that the restriction map induces an isomorphism*

$$j^* :\ \Gamma(X,\mathscr{S})^{\chi^m} \xrightarrow{\sim} \Gamma(X^{ss}_\chi,\ j^*\mathscr{S})^{\chi^m} \qquad \text{for all } m \geq m(\mathscr{S}).$$

(ii) *In the situation of Corollary 11.1.6(ii) the functors ϖ^* and $(\varpi_*(-))^G$ provide mutually inverse equivalences $\mathrm{Coh}(X^{ss}_\chi) \leftrightarrows \mathrm{Coh}^G(X/\!\!/_\chi G)$. Moreover, there is a natural isomorphism*

$$j^*\mathscr{S} \ \cong\ \varpi^* \circ \mathbb{F}\big(\oplus_{m\geq 0}\, \Gamma(X,\mathscr{S})^{\chi^m}\big),$$

of functors from $\mathrm{Coh}^G(X)$ *to* $\mathrm{Coh}^G(X^{ss}_\chi)$.

Here, the first statement in part (ii) is well known. A sketch of proof of other statements may be found in [**?**, §7.4].

12. Appendix 3: Somesse vanishing

12.1. Let $\pi : X \to Y$ be a projective morphism, where X is a smooth and Y is a normal variety. In [K3], Kaledin proves the following result that can also be deduced (with some work) from general vanishing theorems due to Esnault and Viehweg [EV].

Theorem 12.1.1. *For any* $p, q \geq 0$ *such that* $p + q > \dim(X \times_Y X)$, *one has* $R^p\pi\Omega^q_X = 0$.

Let now (X, ω) be a symplectic manifold of dimension $2n$. Then, we have $\Omega^1_X \cong \mathcal{T}_X$; also, the volume form $\wedge^n\omega$ provides a trivialization of the canonical bundle $\mathcal{K}_X = \Omega^{2n}_X$. Hence, one obtains a chain of isomorphisms

$$\Omega^p_X \;\cong\; \wedge^p\mathcal{T}_X \;\cong\; \wedge^p\mathcal{T}_X \;\cong\; \Omega^{\dim X - p}_X,$$

where the last isomorphism is given by contraction of the volume form $\wedge^n\omega$ with p-polyvector fields. Thus, from Theorem 12.1.1, we deduce

Corollary 12.1.2. *Given a smooth symplectic manifold* X *and a projective morphism* $\pi : X \to Y$, *one has*

$$R^p\pi\Omega^q_X = 0, \qquad \forall\, p - q > \dim(X \times_Y X) - \dim X.$$

In the special case where the map π *is a symplectic resolution, we deduce:* $R^p\pi\Omega^q_X = 0$ *whenever* $p > q$.

12.2. **Proof of Theorem 12.1.1.** The argument below, based on Saito's theory of mixed Hodge module was communicated to me by Sasha Beilinson. It is different (and much shorter) than the proof given in [K3].

We begin with some general remarks. Let κ be the image of the canonical element $\mathrm{Id} \in \mathrm{End}(\pi^*\mathcal{T}^*_Y) = \pi^*\mathcal{T}^*_Y \otimes \pi^*\mathcal{T}_Y$ under the morphism

$$d\pi \otimes \mathrm{Id}_{\pi^*\mathcal{T}_Y} : \; \pi^*\mathcal{T}^*_Y \otimes \pi^*\mathcal{T}_Y \;\longrightarrow\; \mathcal{T}^*_X \otimes \pi^*\mathcal{T}_Y.$$

For each pair of integers i, j, let $E^i_j := \wedge^{\dim X + i}\mathcal{T}^*_X \otimes \mathrm{Sym}^{j+i}\,\pi^*\mathcal{T}_Y$. Then, multiplication by κ gives a Koszul type differential $E^i_j \to E^{i+1}_j$. Thus, for each j we obtain a complex (E_j, κ).

The canonical sheaf $\mathcal{K}_X$ has a natural structure of right $\mathcal{D}_X$-module. Furthermore, this $\mathcal{D}_X$-module comes equipped with the structure of a pure Hodge module, with Hodge filtration $F^0\mathcal{K}_X = \mathcal{K}_X$ and $F^1\mathcal{K}_X = 0$. Let $\pi_\dagger(\mathcal{K}_X, F)$ denote a direct image of that module in the derived category of filtered $\mathcal{D}$-modules. Thus, $\pi_\dagger(\mathcal{K}_X, F)$ is a filtered $\mathcal{D}_Y$-complex. So, for any j, there is an associated graded complex $\mathrm{gr}^j_F(\pi_\dagger(\mathcal{K}_X, F))$, of $\mathcal{O}_Y$-modules.

Lemma 12.2.1. *There is a canonical isomorphism* $\mathrm{gr}^j_F(\pi_\dagger(\mathcal{K}_X, F)) \cong R\pi_*E_j$ *for all* $j \in \mathbb{Z}$.

We now proceed with the proof of Theorem 12.1.1. We put $d := \dim(X \times_Y X) - \dim X$.

Step 1. We claim that, for all $n > d$, we have $H^n\pi_\dagger\mathcal{K}_X = 0$.

To prove this, we can replace $\mathcal{D}$-modules by perverse sheaves. Then $\mathcal{K}_{\widetilde{X}}$ becomes $\mathbb{C}_{\widetilde{X}}[\dim \widetilde{X}]$ and we want to check that the top perverse cohomology of $R\pi_*\mathbb{C}_{\widetilde{X}}[\dim \widetilde{X}]$ occurs in degree $\leq d$. Cut X by strata X_i such that $\widetilde{X}_i := \pi^{-1}(X_i) \to X_i$ is a topological fibration. The restriction of $R\pi_*\mathbb{C}_{\widetilde{X}}$ to X_i has locally constant cohomology sheaves that lie in degrees $[0,\ 2(\dim \widetilde{X}_i - \dim X_i)]$. Viewed as perverse sheaves, they lie in degrees

$$[\dim X_i,\ 2(\dim \widetilde{X}_i - \dim X_i) + \dim X_i] \;=\; [\dim X_i,\ \dim(\widetilde{X}_i \times_{X_i} \widetilde{X}_i)].$$

Therefore $R\pi_*\mathbb{C}_{\widetilde{X}}[\dim \widetilde{X}]$ has a filtration whose successive quotients are !-extensions from X_i to X of complexes of perverse sheaves with top cohomology in degree $\dim(\widetilde{X}_i \times_{X_i} \widetilde{X}_i) - \dim \widetilde{X}$. The !-extension is right t-exact (for perverse t-structure), and $\dim(\widetilde{X}_i \times_{X_i} \widetilde{X}_i)$ is not larger than $\dim(\widetilde{X} \times_X \widetilde{X})$. We are done.

Step 2. Saito's theory of polarizable Hodge modules insures that the Hodge filtration on the complex $\pi_\dagger(\mathcal{K}_X, F)$ is strictly compatible with the differential, i.e. one has $H^n \operatorname{gr}_F \pi_\dagger(\mathcal{K}_X, F) = \operatorname{gr}_F H^n \pi_\dagger(\mathcal{K}_X, F)$. Hence, using Lemma 12.2.1, we deduce

$$R^n\pi_* E_j \cong H^n(\operatorname{gr}^j_F(\pi_\dagger(\mathcal{K}_X, F))) \cong \operatorname{gr}^j_F H^n \pi_\dagger(\mathcal{K}_X, F) = 0, \qquad \forall n > d. \tag{12.2.2}$$

Observe next that the complex $R\pi_* E_j$ has a natural increasing filtration, $G_\bullet R\pi_* E_j$, such that $\operatorname{gr}^G_i R\pi_* E_j = R\pi_*(\wedge^{\dim X - i}\mathcal{T}^*_X \otimes \operatorname{Sym}^{j-i} \pi^*\mathcal{T}_Y)[i]$. Thus, there are short exact sequences

$$0 \to G_{j-1}E_j \to E_j \to R\pi_* \wedge^{\dim X - j} \mathcal{T}^*_X[j] \to 0. \tag{12.2.3}$$

Step 3. We claim that $H^n G_{j-1}E_j = 0$ for all $n > d$. We prove this by downward induction on $q := \dim X - j$ using (12.2.2) as the base of induction. To prove the induction step, suppose for some j we have shown that $H^n G_{j-1}E_j = 0$ for all $n > d$. Then, for $q = \dim X - j$, from (12.2.3) we deduce $H^n(R\pi_* \wedge^q \mathcal{T}^*_X[j]) = R^{n+\dim X - q}\pi_* \wedge^q \mathcal{T}^*_X = 0$. This proves the claim.

Now, our claim implies that for $p + q > d + \dim X$, we have $R^p\pi_* \wedge^q \mathcal{T}^*_X = 0$. ☐

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA
E-mail address: `ginzburg@math.uchicago.edu`