The Plateau Problem

Definition

Let $M$ be an $m$-dimensional surface in $\mathbb{R}^N$ with boundary $\Gamma$. We say that $M$ is a least-area surface if its area is $\leq$ the area of any other surface having the same boundary.

Remark: To make this definition precise, one has to specify the class of surfaces allowed. (E.g., do we require the surfaces to be oriented?)

Plateau problem: Given a boundary $\Gamma$, does there exist a least-area surface $M$ with boundary $\Gamma$? How smooth is $M$?

Easy example: $m = 1$. of the Plateau problem is the straight line segment joining them.
In general, even proving existence is very nontrivial. In fact, in 1936 Jesse Douglas won the first Fields medal for his existence and regularity theorems for the $m = 2$ case of the Plateau problem. I will discuss those results on Friday.

Now, I want to consider a related question: given a surface $M$, how do we tell if it a least-area surface? In general, it's very hard to tell, but there is a “first-derivative test” that provides a necessary condition in order for $M$ to be a least-area surface:

If $M_t$ is a one-parameter family of surfaces each with boundary $\Gamma$, and if $M_0 = M$, then

\[
\left( \frac{d}{dt} \right)_{t=0} \text{area}(M_t)
\]

should be 0.
Let $M$ be a compact $m$-dimensional manifold in $\mathbb{R}^N$. Let $\phi_t : M \to \mathbb{R}^N$ be a smooth one-parameter family of smooth maps such that $\phi_0(p) \equiv p$.

Let $X(p) = \left(\frac{d}{dt}\right)_{t=0} \phi_t(p)$ be the initial velocity vectorfield

$$\phi_t(p) = p + t X(p) + o(t).$$
Theorem

\[
\left( \frac{d}{dt} \right)_{t=0} \text{area}(\phi_t M) = \int_M \text{div}_M(X) \, dS \\
= \int_{\partial M} X \cdot \nu_{\partial M} \, ds - \int_M H \cdot X \, dS.
\]

Here

\[
\text{div}_M X = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i} X,
\]

\[H(p) = \text{the mean curvature vector of } M \text{ at } p.\]

\[e_1(p), \ldots, e_m(p) \text{ is an orthonormal basis for } \text{Tan}_p M.\]
Perhaps it would be more accurate refer to the first equality

\[
\left( \frac{d}{dt} \right)_{t=0} \text{area}(\phi_t M) = \int_M \text{div}_M(X) \, dS,
\]

as the “first variation formula” and to refer to the second equality

\[
\int_M \text{div}_M X \, dS = \int_{\partial M} X \cdot \nu_{\partial M} \, ds - \int_M H \cdot X \, dS
\]

as the “generalized divergence theorem”. (Note that when the vectorfield \(X\) is tangent to \(M\), it is just the ordinary divergence theorem.)
**First equality:** Note that

\[
\text{area}(\phi_t M) = \int J_m(D\phi_t) \, dS
\]

so

\[
\left( \frac{d}{dt} \right)_{t=0} \text{area}(\phi_t M) = \int \left( \frac{d}{dt} \right)_{t=0} J_m(D\phi_t) \, dS.
\]

Thus it suffices to calculate that

\[
\left( \frac{d}{dt} \right)_{t=0} J_m(D\phi_t) = \text{div}_M X.
\]

I’ll give the details later if time permits.
Proof of 2nd equality

\[ \int_M \text{div}_M X \, dS = \int_M \text{div}_M (X^T) \, dS + \int_M \text{div}_M (X^N) \, dS \]

\[ = \int_{\partial M} X^T \cdot \nu_{\partial M} \, ds + \int_M \text{div}_M (X^N) \, dS \]

\[ = \int_{\partial M} X \cdot \nu_{\partial M} \, ds + \int_M \text{div}_M (X^N) \, dS. \]

So we just have to show that \( \text{div}_M (X^N) = -H \cdot X \).
Using summation convention,

\[
\text{div}_M(X^N) = e_i \cdot \nabla e_i(X^N)
\]

\[
= \nabla e_i(e_i \cdot X^N) - (\nabla e_i e_i) \cdot X^N
\]

\[
= \nabla e_i(0) - (\nabla e_i e_i)^N \cdot X
\]

\[
= -H \cdot X.
\]

**Definition**

An \(m\)-dimensional submanifold \(M \subset \mathbb{R}^N\) (or of a Riemannian manifold) is called **minimal** provided \(H \equiv 0\), i.e., provided it is a critical point for the area functional.
Theorem

Let $M$ be a compact $m$-dimensional minimal submanifold of $\mathbb{R}^N$. Then

$$m \text{ area}(M) = \int_{x \in \partial M} x \cdot \nu_{\partial M} \, ds. \quad (*)$$

Proof.

We apply

$$\int_M \text{div}_M X \, dS = \int_{\partial M} X \cdot \nu_{\partial M} - \int_M X \cdot H \, dS$$

to the vectorfield $X(x) \equiv x$. Then $\text{div}_M X \equiv m$ and $H \equiv 0$, so we get $(*)$. \qed
The monotonicity formula

**Theorem**

Let $M$ be a minimal submanifold of $\mathbb{R}^N$ and let $p \in \mathbb{R}^N$. Then

$$\Theta(M, p, r) := \frac{\text{area}(M \cap B(p, r))}{\omega_m r^m}$$

is an increasing function of $r$ for $0 < r \leq R := \text{dist}(p, \partial M)$. Indeed,

$$\left(\frac{d}{dt}\right) \Theta(M, p, r) \geq 0$$

with equality if and only if $M$ intersects $\partial B(p, r)$ orthogonally.
Proof of monotonicity \((m = 2)\)

We may assume \(p = 0\). Let \(M_r = M \cap B(0, r)\),

Let \(A(r) = \text{area}(M_r)\) and \(L(r) = \text{length}(\partial M_r)\).

Then
\[
A'(r) \geq L(r).
\]

Also, by the previous theorem,

\[
2A(r) \leq \int_{\partial M_r} x \cdot \nu_{\partial M_r} \, ds \leq rL(r).
\]

So
\[
A' - 2r^{-1}A \geq 0.
\]
\[
r^{-2}A' - 2r^{-3}A \geq 0.
\]
\[
(r^{-2}A)' \geq 0. \qed
\]
Note that if $M$ is a smooth immersed surface and if $p \in M \setminus \partial M$, then

$$\Theta(M, p) := \lim_{r \to 0} \Theta(M, p, r) = \text{number of sheets of } M \text{ passing through } p.$$ 

In particular, $\Theta(M, p) \geq 1$. 
Let $M$ be a properly immersed minimal surface (without boundary) in $\mathbb{R}^N$. Then $\Theta(M, p, r)$ is increasing for $0 < r < \infty$. Thus $\lim_{r \to \infty} \Theta(M, p, r)$ exists.

We denote the limit by $\Theta(M)$ and call it the “density of $M$ at infinity”.
Density at infinity is 2. (Picture courtesy of Matthias Weber.)
Density at infinity is 2. (Picture courtesy of Matthias Weber.)
Uniqueness of the plane

**Theorem**

Let $M$ be a properly immersed minimal $m$-manifold without boundary in $\mathbb{R}^N$. Then $\Theta(M) \geq 1$ with equality if and only if $M$ is a multiplicity 1 plane.

**Proof.**

Let $p \in M$. Then by monotonicity,

$$1 \leq \Theta(M, p, r) \leq \Theta(M) \quad (*)$$

This proves the inequality. If $1 = \Theta(M)$, then we would have equality in (*), so $M$ would intersect $\partial B(p, r)$ orthogonally for every $r$. That implies that $M$ is invariant under dilations about $p$, which implies that $M = \text{Tan}_p M$. \qed
Details of proof of first variation formula

Recall

\[ \phi_t(p) = p + t \, X(p) + o(t). \]
Now

\[ J_m(D\phi_t) = \text{area of parallelepiped given by } D\phi_t(e_1), \ldots, D\phi_t(e_m) \]

\[ = \sqrt{\det(D\phi_t(e_i) \cdot D\phi_t(e_j))} \]

Now \( D\phi_t(e_i) = \nabla_{e_i}(\phi_t) \approx \nabla_{e_i}(p + tX(p)) = e_i + t\nabla_{e_i}X. \)

Here \( \approx \) means “differ by \( o(t) \).”
Thus

\[ J_m(D\phi_t) \cong \sqrt{\det((e_i + t\nabla_{e_i}X) \cdot (e_j + t\nabla_{e_j}X))} \]

\[ \cong \sqrt{\det(\delta_{ij} + t(e_i \cdot \nabla_{e_j}X + e_j \cdot \nabla_{e_i}X))} \]

Recall that \( \det(I + tA) = 1 + t \text{trace}(A) + o(t) \), so

\[ J_m(D\phi_t) \cong \sqrt{1 + 2t \sum_i (e_i \cdot \nabla_{e_i}X)} \]

\[ \cong \sqrt{1 + 2t \text{div}_M X} \cong 1 + t \text{div}_M X. \]
Recall the monotonicity theorem: If $M$ is minimal, then the density ratio
\[ \Theta(M, p, r) = \frac{\text{area}(M \cap B(p, r))}{\omega_m r^m} \]
is an increasing function of $r$ for $0 < r < R = \text{dist}(p, \partial M)$.

The assumption $r < \text{dist}(p, \partial M)$. For example, if $M \subset B(p, R)$ with $\partial M \subset \partial B(p, R)$, then $\Theta(M, p, r)$ is strictly decreasing for $r \geq R$.

However, there is an extension of the monotonicity theorem that gives information for all $r$: 
**Theorem**

Let $M \subset \mathbb{R}^N$ be a compact, minimal $m$-manifold with boundary $\Gamma$, and let $p \in \mathbb{R}^N$. Let $E = E(p, \Gamma)$ be the exterior cone with vertex $p$ over $\Gamma$:

$$E = \bigcup_{q \in \Gamma} \{ p + t(q - p) : t \geq 1 \}.$$  

Let $\tilde{M} = M \cup E$. Then

$$\Theta(\tilde{M}, p, r) := \frac{\text{area}(\tilde{M} \cap B(p, r))}{\omega_m r^m}$$

is an increasing function of $r$ for all $r > 0$.

**Proof:** We may assume $p = 0$. As before, we will apply the first variation formula (generalized divergence theorem) to the vectorfield $X(x) = x$. 
Lemma

Let $E$ be the exterior cone over $\Gamma$ with vertex $0$. Among all unit vectors $v$ that are normal to $\Gamma$ at $x \in \Gamma$, the minimum value of $x \cdot v$ is attained by $v = -\nu_{\partial E}(x)$.

Consequently, $x \cdot (v + \nu_{\partial E}(x)) \leq 0$ for any such vector $v$.

Proof of the lemma is left as an exercise.
Let $M_r$, $E_r$, $\tilde{M}_r$, and $\Gamma_r$ be the portions of $M$, $E$, $\tilde{M}$, and $\Gamma$ inside the ball $B_r = B(0, r)$. By the generalized divergence theorem,

$$
\int_{M_r} \text{div}_M X \, dS = \int_{\partial M_r} X \cdot \nu_{\partial M_r} \, ds - \int_{M_r} H \cdot X \, dS = \int_{\partial M_r} X \cdot \nu_{\partial M_r} \, ds
$$

since $H \equiv 0$ on $M$. Similarly,

$$
\int_{E_r} \text{div}_M X \, dS = \int_{\partial E_r} X \cdot \nu_{\partial E_r} \, ds - \int_{E_r} H \cdot X \, dS = \int_{\partial M_r} X \cdot \nu_{\partial M_r} \, ds
$$

because $H \cdot X \equiv 0$ on $E$, since $H$ is perpendicular to $E$ and $X$ is tangent to $E$. 
Also, $\text{div}_M X \equiv \text{div}_E X \equiv m$, so the left sides of these equations are $m \text{ area}(M_r)$ and $m \text{ area}(E_r)$.

Adding these the two equations gives

$$m \text{ area}(\tilde{M}_r) \leq \int_{\partial M_r} x \cdot \nu_{\partial M_r} \, dS + \int_{\partial E_r} x \cdot \nu_{\partial E_r} \, dS. \quad (*)$$

Note that $\partial M_r$ consists of two parts: $M \cap \partial B_r$ and $\Gamma_r$. Likewise $\partial E_r$ consist of $E \cap \partial B_r$ and $\Gamma_r$. Thus we can rewrite (*) as

$$m \text{ area}(\tilde{M}_r) \leq \int_{\partial \tilde{M}_r} x \cdot \nu_{\partial \tilde{M}_r} \, ds + \int_{\Gamma_r} x \cdot (\nu_{\partial M_r} + \nu_{\partial E_r}) \, ds. \quad (\dagger)$$

By the lemma, the second integrand is everywhere $\leq 0$. Thus

$$m \text{ area}(\tilde{M}_r) \leq \int_{x \in \partial \tilde{M}_r} x \cdot \nu_{\partial \tilde{M}_r} \, ds.$$

The rest of the proof is exactly the same of the proof of the monotonicity theorem.