Introduction to Minimal Surface Theory: Lecture 2

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By a theorem of Morrey, every surface admits local isothermal coordinates (i.e., can be parametrized locally by conformal maps from domains in $\mathbb{R}^2$.)

**Theorem**

Let $F : \Omega \in \mathbb{R}^2 \rightarrow \mathbb{R}^N$ be a conformal immersion. Then $F(\Omega)$ is minimal if and only if $F$ is harmonic.

**Proof.**

One way to show this is to calculate that $H$ is equal to the $\Delta_g F$, where $g$ is the metric on $M$ induced by the metric on $\mathbb{R}^N$. (This is also true for $m$-dimensional $M$ in general Riemannian manifolds.) Thus $M$ is minimal if and only if $F$ is harmonic with respect to the metric $g$. Also, for two dimensional surfaces, harmonic functions remain harmonic under conformal change of domain.
Analyticity and the convex hull property

Corollary

*Every two-dimensional $C^2$ minimal surface in $\mathbb{R}^n$ is real-analytic.*

Corollary (Convex hull property)

*If $M$ is a compact 2d minimal surface in $\mathbb{R}^n$, then $M$ lies in the convex hull of $\partial M$."

Proof.

For simply connected $M$, parametrize $M$ by a conformal harmonic map $F : D \to \mathbb{R}^n$. If $L : \mathbb{R}^n \to \mathbb{R}$ is linear, then $L \circ F$ is harmonic so it achieves its maximum on $\partial D$. Thus $L|_M$ achieves it maximum on $\partial D$.

These corollaries are also true for $m$-dimensional $C^2$ minimal submanifolds of $\mathbb{R}^n$ (by different proofs.)
Theorem

Let $M$ be a minimal surface in $\mathbb{R}^3$. Then $M$ is minimal if and only if the Gauss map $\mathbf{n} : M \to S^2$ is (almost) conformal and orientation-reversing.

Proof.

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the principal directions of $M$ at $p \in M$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ form an orthonormal basis for $\text{Tan}_p M$ and also for $\text{Tan}_{\mathbf{n}(p)} S^2$. With respect to this basis, the matrix for $D\mathbf{n}(p)$ is

$$
\begin{bmatrix}
\kappa_1 & 0 \\
0 & \kappa_2
\end{bmatrix}
$$
For any surface $M \subset \mathbb{R}^3$, the Gauss curvature $K = \kappa_1 \kappa_2$ is the signed Jacobian of the Gauss map. If $M$ is minimal, $K \leq 0$, so

$$- \int_M K \, dS = \int_M |K| \, dS = \text{area}(n(M))$$

Here area is counted with multiplicity:

$$\text{area}(n(M)) = \int_{p \in S^2} \#n^{-1}(p) \, dp.$$
Theorem (Osserman)

Let $M \subset \mathbb{R}^3$ be a complete orientable minimal surface of finite total curvature:

$$TC(M) = \int_M |K| \, dS = \int_M (-K) \, dS < \infty.$$  

Then

1. $M$ is conformally equivalent to a compact Riemann surface minus finitely many points:

   $$M \cong \Sigma \setminus \{p_1, \ldots, p_k\}.$$

2. The Gauss map extends to the punctures.

3. The total curvature of $M$ is an integral multiple of $4\pi$.

4. $M$ is proper.
The Gauss map is a conformal diffeomorphism from $M$ to $S^2 \setminus \{NP, SP\}$. The Gauss map extends continuously to the punctures. The total curvature is the area of the Gaussian image, namely $4\pi$. 
Proof

The first assertion is a special case of an intrinsic theorem due to Huber: If $M$ is a complete surface such that

$$\int_M K^- dS < \infty$$

then $M$ is conformally a punctured Riemann surface. ($K^- = (|K| - K)/2$.)

Second assertion: Let $U \subset \Sigma$ be a neighborhood of one of the punctures, $p$. By Picard, either $n : U \setminus \{p\} \to S^2 \cong \mathbb{C} \cup \{\infty\}$ is meromorphic at $p$ (and therefore extends to $p$), or $n : U \setminus \{p\} \to S^2$ takes all but 3 values in $S^2$ infinitely many times.

The latter implies that $\int_U |K| dS = \infty$, a contradiction.

Thus $n$ extends continuously (indeed smoothly) to $U$. 
Third assertion: By the second assertion, $n$ extends to a map $n : \Sigma \to S^2$. The signed jacobian determinant is everywhere $\leq 0$ (and $< 0$ except at isolated points).

Thus for almost all $p \in S^2$, the number of pre images of $p$ under $n$ is equal to $|d|$, where $d$ is the mapping degree.

Hence

$$
\int_M |K| \, dS = \int_M (-K) \, dS = \int_{S^2} \#n^{-1}(\cdot) \, dS = |d| \cdot 4\pi.
$$
Fourth assertion: properness.

This can be proved using the Weierstrass representation (discussed later in this lecture.)

Alternatively, one can show that if $S \subset \mathbb{R}^3$ is a complete surface diffeomorphic to a closed disk minus its center and if the slope of $\text{Tan}(S, p)$ is uniformly bounded, then $S$ is proper in $\mathbb{R}^3$. (See White, “Complete Surfaces of Finite Total Curvature”, JDG 26 (1987).) One applies this result to a small neighborhood in $\Sigma$ of a puncture, on which one can assume (by rotating) that the unit normal is very nearly vertical.
Another characterization of the plane

**Corollary**

If $M \subset \mathbb{R}^3$ is a complete, orientable minimal surface of total curvature $< 4\pi$, then $M$ is a plane.

**Proof.**

If the total curvature is less than $4\pi$, it must be 0, so $K \equiv 0$. But for a minimal surface, $K(p) = 0$ implies that the principle curvatures at $p$ are 0.

We’ll see tomorrow that this corollary implies a very useful curvature estimate for minimal surfaces.
We saw that minimal surfaces in $\mathbb{R}^3$ are precisely those that can be parametrized (locally) by conformal, harmonic maps $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$.

Weierstrass found a nice way to generate all such $F$. 
Write \( z = x + iy \) in \( \mathbb{R}^2 \), and

\[
\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad , \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).
\]

Note that

\[
\partial_z \partial_{\bar{z}} = \frac{1}{4}(\partial_x + i\partial_y)(\partial_x - i\partial_y) = \frac{1}{4}(\partial_x^2 + \partial_y^2) = \frac{1}{4}\Delta.
\]

Thus

\[
F \text{ is harmonic } \iff F_{zz} \equiv 0 \iff F_z \text{ is holomorphic}.
\]
What about conformality?

\[ F_z \cdot F_z = \frac{1}{4} (F_x - iF_y) \cdot (F_x - iF_y) \]
\[ = \frac{1}{4} (|F_x|^2 - |F_y|^2 - 2iF_x \cdot F_y) . \]

So

\[ \Re(F_z \cdot F_z) = |F_x|^2 - |F_y|^2 \quad \Im(F_z \cdot F_z) = 2F_x \cdot F_y. \]

Consequently, \( F \) is conformal if and only if \( F_z \cdot F_z \equiv 0 \).
Let $\phi = F_z = (X_1, X_2, X_3) \in \mathbb{C}^3$. Our goal is to find holomorphic $\phi$ such that $\phi \cdot \phi \equiv 0$. We then recover $F$ by

$$F = \Re \left( \int F_z \, dz \right) = \Re \left( \int \phi \, dz \right).$$
Note that $\phi(z) = F_z = \frac{1}{2}(F_x - iF_y)$ determines the tangent plane to $M$ at $F(z)$, and therefore the image $n(z)$ of $F(z)$ under the Gauss map, and therefore the image $g(z) \in \mathbb{R}^2 \cong \mathbb{C}$ of $n(z)$ under stereographic projection.

Indeed, one can check that

$$g(z) = \frac{X_3}{X_1 - iX_2}.$$

Let $f(z) = X_1 - iX_2$.

Then $X_3 = gf$,

and one can also solve for $X_1$ and $X_2$ in terms of $f$ and $g$:

$$X_1 = \frac{1}{2}(1 - g^2)f,$$

$$X_2 = \frac{i}{2}(1 + g^2)f.$$
Theorem

Let $\Omega \subset \mathbb{C}$ be simply connected and let $f$ be holomorphic and $g$ be meromorphic on $\Omega$. Then

$$F = \Re \int \left( \frac{1}{2}(1 - g^2), \frac{i}{2}(1 + g^2), g \right) f \, dz$$

(*)

is a harmonic, conformal mapping of $\Omega$ into $\mathbb{R}^3$. Furthermore, every harmonic, conformal map $F : \Omega \to \mathbb{R}^3$ arises in this way.

(I’ve omitted a technical condition: wherever $g$ has a pole of order $m$, $f$ should have a zero of order $2m$.)

More generally, $f \, dz$ and $g$ can be any holomorphic differential and any meromorphic function on any Riemann surface $\Omega$. But if $\Omega$ is not simply connected, (*) may only be well-defined on the universal cover. (“Period problem”).
To be precise, the Weierstrass representation (*) gives an immersion of \( \Omega \) (rather than of its universal cover) if and only if the closed one forms

\[
\frac{1}{2}(1 - g^2)f \, dz, \quad \frac{i}{2}(1 + g^2)f \, dz, \quad \text{and} \quad gf \, dz
\]

have no real periods.
Example: Enneper’s Surface

\[ f = 1 \text{ and } g = z \]
Examples: Catenoid and Helicoid

\[ f = \frac{1}{z}, \quad g = z, \quad z \neq 0 \]

\[ f = 1, \quad g = e^{iz} \]
Since $f \, dz$ and $g$ determine the surface, all geometric quantities can be expressed in terms of $f$ and $g$. For example, the conformal factor is

$$
\lambda = \frac{(1 + |g|^2)}{2} |f|
$$

i.e.,

$$
\begin{align*}
\frac{ds^2}{|dz|^2} &= \left[ \frac{(1 + |g|^2)}{2} \right]^2 |f|^2 |dz|^2
\end{align*}
$$

and the Gauss curvature is

$$
K = -\left[ \frac{4|g'|}{|f|(1 + |g|^2)^2} \right]^2.
$$
A surface $M \subset \mathbb{R}^3$ is **flexible** if it can be deformed (non-trivially) through a one-parameter family of isometric immersions.

Example: plane. (This is why you can bend a sheet of paper.)

Theorem: Every smooth, closed convex hypersurface in $\mathbb{R}^N$ is rigid.

Open question: is there a smooth, closed, flexible surface in $\mathbb{R}^3$?
Can a surface (not necessarily closed) in $\mathbb{R}^3$ be flexed (isometrically) in such a way that normal vectors don’t change? In other words, during the isometric deformation, as each point moves, the unit normal at that point should remain the same. It seems impossible, but...
The answer is yes for minimal surfaces!

In the Weierstrass representation, replace $f$ by $e^{i\theta} f$.

The metric only depends on $|g|$ and $|f|$, so doesn’t change, and the Gauss map $g$ doesn’t change.
For example, if we start with the helicoid, this gives an isometric deformation of the helicoid to the (universal cover of) the catenoid.
Weierstrass data: $f = 1$ and $g = z$.

Since conformal factor only depends on $|f| \equiv 1$ and $|g| = |z|$, the surface is intrinsically rotationally symmetric about 0 (which is not obvious from looking at pictures.)