In many situations, one wants to take limits of minimal surfaces.

For example, for several centuries, the plane and the helicoid were the only known complete, properly embedded minimal surfaces in $\mathbb{R}^3$ with finite genus and with exactly one end.

Jacob Bernstein and Christine Breiner proved (using work of Colding and Minicozzi) that any such surface (other than a plane) must asymptotic to a helicoid at infinity. Hence such a surface of genus $g$ is called a genus-$g$ helicoid.

But do genus-$g$ helicoids exist for $g \neq 0$?
In 2004 (publ. 2009), Hoffman, Weber, and Wolf proved (using the Weierstrass representation) existence of a genus-1 helicoid:

What about genus $> 1$?
This year, David Hoffman, Martin Traizet, and I proved existence of genus $g$ helicoids for every $g$:

![Helicoid Image]

Picture by Traizet 1993 (!).
In our proof, first we construct analogous surfaces in $S^2 \times \mathbb{R}$ (which turns out to be easier!), and then we get examples in $\mathbb{R}^3$ by letting the radius of $S^2$ tend to infinity.

Of course we need to know that the surfaces in $S^2 \times \mathbb{R}$ converge smoothly to limit surfaces in $\mathbb{R}^3$. 
In general, it is very useful to have compactness theorems: conditions on a sequence of minimal surfaces that guarantee existence of a smoothly converging subsequence.
Theorem

Let $M_i$ be a sequence of minimal submanifolds of $\mathbb{R}^N$ (or of a Riemannian manifold) such that second fundamental forms are uniformly bounded. Then (locally) there exist smoothly converging subsequences.

In particular, if $p_i \in M_i$ is a sequence bounded in $\mathbb{R}^N$ and if $\text{dist}(p_i, \partial M_i) \geq R > 0$, then (after passing to a subsequence),

$$M_i \cap B(p_i, \partial M_i)$$

converges smoothly to a limit minimal surface $M^*$. 

Here dist is intrinsic distance in $M_i$, and $B(p_i, r)$ is the geodesic ball of radius $r$ in $M_i$.

Note: $B$ is the open ball.
The Extrinsic Version

The theorem (if interpreted properly) is also true when dist is exterior distance and $\mathbf{B}(p, r)$ is the extrinsic ball.

In this case the conclusion is that there is an $r$ with $0 < r < R$ such that the connected component of $M_i \cap \mathbf{B}(p_i, r)$ containing $p_i$ converges smoothly (after passing to a subsequence) to a limit surface $M^\ast$. 
Let us assume that the principle curvatures are bounded by 1, and that $\text{dist}(p_i, \partial M_i) \geq \pi/2$.

We can assume the $p_i$ converge to a limit $p$ and that $\text{Tan}(M_i, p_i)$ converge to a limit plane. Indeed, in $\mathbb{R}^n$, we can assume by translating and rotating that $p_i \equiv 0$ and that $\text{Tan}(M_i, p_i)$ is the horizontal plane through 0.

For each $i$, let $S_i$ be the connected component of

$$M_i \cap (B^m(0, 1/2) \times \mathbb{R}^{N-m})$$

containing 0.

The hypotheses imply that $S_i$ is the graph of a function

$$F_i : B^m(0, 1/2) \rightarrow \mathbb{R}^{N-m}$$

where the $C^2$ norm of the $F_i$ is uniformly bounded.
Hence by Arzela-Ascoli, we may assume (by passing to a subsequence) that the $F_i$ converge in $C^{1,\alpha}$ to a limit function $F$.

So far we have not used minimality.

Since the surfaces $M_i$ are minimal, the $F_i$ are solutions to an elliptic partial differential equation (the minimal surface equation).

According the theory of such equations, convergence in $C^{1,\alpha}$ on $B^m(0, \frac{1}{2})$ implies convergence in $C^k$ on $B^m(0, \frac{1}{2} - \epsilon)$ (for any $k$ and $\epsilon$).
This theorem indicates the importance of curvature estimates: curvature estimates for a classes of minimal surfaces imply existence of smooth subsequential limits of such surfaces.
Theorem (w1987)

For every \( \lambda < 4\pi \), there is a \( C < \infty \) with the following property. If \( M \subset \mathbb{R}^3 \) is an orientable minimal surface with total curvature \( \leq \lambda \), then

\[
|A(p)| \ dist_M(p, \partial M) \leq C.
\]

The theorem is false for \( \lambda = 4\pi \), since the catenoid has total curvature \( 4\pi \) and is not flat.

(Cf. Choi-Schoen 1985)
Proof: It suffices to prove it when $M$ is a smooth, compact manifold with boundary.

(A general surface can be exhausted by such $M$.)

Suppose the theorem is false. Then there is a sequence $p_i \in M_i$ of examples with

$$TC(M_i) \leq \lambda$$

and

$$|A_i(p_i)| \text{ dist}(p_i, \partial M_i) \to \infty.$$

We may assume that each $p_i$ has been chosen in $M_i$ to maximize the left side of $(*)$.

By translating and scaling, we may assume that $p_i = 0$ and that $|A_i(p_i)| = 1$, and therefore that $\text{dist}(0, \partial M_i) \to \infty$. 
We may also replace $M_i$ by the geodesic ball of radius $R_i := \text{dist}(0, \partial M_i)$ about 0.

We have:

$$|A_i(0)| \equiv 1,$$

$$R_i = \text{dist}(0, \partial M_i) \to \infty,$$

and

$$|A_i(x)| \text{ dist}(x, \partial M_i) \leq \text{dist}(0, \partial M_i).$$

Now $\text{dist}(0, x) + \text{dist}(x, \partial M_i) = \text{dist}(0, \partial M_i) = R_i$, so

$$|A_i(x)| \leq \frac{R_i}{\text{dist}(x, \partial M_i)} = \frac{R_i}{R_i - \text{dist}(0, x)} \leq \frac{R_i}{R_i - r}$$

if $\text{dist}(x, \partial M_i) \leq r$. 
We have shown for each $r$ that

$$\sup_{\text{dist}(x,0) \leq r} |A_i(x)| \leq \frac{R_i}{R_i - r} \to 1.$$ 

Hence the $M_i$ converge smoothly (first compactness theorem) to a complete minimal surface $M$ with $|A_M(0)| = 1$.

Thus by the corollary to Osserman’s Theorem, $TC(M) \geq 4\pi$.

However,

$$TC(M) \leq \liminf_i TC(M_i) \leq \lambda < 4\pi. \quad \square$$
1. The theorem is also true (with the same proof) in $\mathbb{R}^n$, but with $4\pi$ replaced by $2\pi$. (This is because Osserman’s theorem is also true in $\mathbb{R}^n$, but with $4\pi$ replaced by $2\pi$.)

2. The same proof gives a version of theorem for minimal surfaces in Riemannian manifolds. (Note that if the curvature of $M_i$ at $p_i$ is blowing up, then we dilate to make it 1, which makes the ambient manifold flatter and flatter. Thus in the limit we get a minimal surface in Euclidean space.)
Recall that we have proved:

1. A complete, nonflat minimal surface in $\mathbb{R}^3$ has total curvature $\geq 4\pi$.

2. For any minimal $M \subset \mathbb{R}^3$ with $TC(M) \leq \lambda < 4\pi$,

$$|A(p)| \operatorname{dist}_M(p, \partial M) \leq C\lambda.$$

We deduced 2 from 1. But conversely, 2 implies 1: if the $M$ in 2 is complete, then $\operatorname{dist}(p, \partial M) = \infty$, so $|A(p)| = 0$. Thus 1 and 2 may be regarded as global and local versions of the same fact.
The equivalence of statements 1 and 2 is an example of general principle: any “Bernstein-type” theorem (i.e., a theorem asserting that certain complete minimal surfaces must be flat) should be equivalent to a local curvature estimate.
Example: the easy version of Allard’s Regularity Theorem

* 
1. Global theorem: If \( M \subset \mathbb{R}^n \) is a proper minimal submanifold without boundary and if \( \Theta(M) \leq 1 \), then \( M \) is a plane.

2. Local estimate: there exist \( \lambda > 1, \epsilon > 0, \) and \( C < \infty \) with the following property. If \( M \subset \mathbb{R}^N \) is minimal, \( \text{dist}(p, \partial M) \geq R \), and \( \Theta(M, p, R) \leq \lambda \), then

\[
\sup_{x \in B(p, \epsilon R)} |A(q)| R \leq C.
\]

(Also true in Riemannian manifolds.)

Clearly \( 2 \implies 1 \), and proof that \( 1 \implies 2 \) is very similar to the proof of the \( TC(M) < 4\pi \) curvature estimate.
Allard’s theorem is much more powerful because he does not assume that $M$ is smooth: it can be any minimal variety (“stationary integral varifold”). He concludes that $M \cap B(p, \varepsilon R)$ is smooth (with estimates).
Concentration Theorem (w1987)

Suppose that $M_i \in \Omega \subset \mathbb{R}^n$ are 2-dimensional minimal surfaces, that $\partial M_i \subset \partial \Omega$, and that $TC(M_i) \leq \Lambda < \infty$.

Then (after passing to a subsequence) there is a set $S \subset \Omega$ of at most $\frac{\Lambda}{2\pi}$ points such that $M_i$ converges smoothly in $\Omega \setminus S$ to a limit minimal surface $M$.

Now suppose $\Omega \subset \mathbb{R}^3$. Then $|S| \leq \frac{\Lambda}{4\pi}$. Also, if the $M_i$ are embedded, then $M \cup S$ is a smooth embedded surface (with multiplicity.)

The theorem remains true (with essentially the same proof) in Riemannian manifolds.
Let $M_n$ be obtained by dilating the catenoid by $1/n$. Then $M_n$ converges to the plane (with multiplicity 2), and the convergence is smooth except at the origin.
Yes:

Suppose the $M_i$ all have the same finite topological type.

Suppose also that the boundary curves $\partial M_i$ are reasonably well-behaved:

$$\sup_i \int_{\partial M_i} |\kappa_{\partial M_i}| \, ds < \infty.$$  

Then we get $\sup_i TC(M_i) < \infty$ by Gauss-Bonnet.
Proof of the concentration theorem in $\mathbb{R}^3$: Define measures $\mu_i$ on $\Omega$ by

$$\mu_i(U) = TC(M_i \cap U).$$

By passing to a subsequence, we can assume that the $\mu_i$ converge weakly to a limit measure $\mu$ with $\mu(\Omega) \leq \Lambda$.

Let $S$ be the set of points $p$ such that $\mu\{p\} \geq 4\pi$. Then $|S| \leq \frac{\Lambda}{4\pi}$.

Suppose $x \in \Omega \setminus S$. Then $\mu\{x\} < \lambda < 4\pi$ for some $\lambda$. Thus there is a closed ball $B = B(x, r) \subset \Omega$ with $\mu(B) < \lambda$. Hence

$$TC(M_i \cap B) = \mu_i(B) < \lambda$$

for all sufficiently large $i$.

But then $|A_i(\cdot)|$ is uniformly bounded on $B(x, r/2)$.

Summary: $|A_i(\cdot)|$ is locally uniformly bounded in $\Omega \setminus S$. Therefore we get the subsequential convergence. 

(I don’t have time to prove the other assertions of the theorem.)
Suppose that $M_i \subset \Omega \subset \text{a Riemannian manifold}$ are minimal surfaces and that $\partial M_i \subset \partial \Omega$. Suppose also that

$$\sup_i \text{genus}(M_i) < \infty$$

and that

$$\sup_i \text{area}(M_i \cap U) < \infty \quad \text{for } U \subset \subset \Omega.$$ 

Then

$$\sup_i TC(M_i \cap U) < \infty \quad \text{for } U \subset \subset \Omega.$$ 

Thus under the hypotheses of this theorem, we get the conclusion of the concentration theorem: smooth converge (after passing to a subsequence) away from a discrete set $S$. 

Theorem
Let $M$ be a compact minimal submanifold of a Riemannian manifold. We say that $M$ is **stable** provided

$$
\left( \frac{d}{dt} \right)^2_{t=0} \text{area}(\phi_t M) \geq 0
$$

for all deformations $\phi_t$ with $\phi_0(x) \equiv x$ and $\phi_t(y) \equiv y$ for $y \in \partial M$.

For $M$ noncompact, we say that $M$ is stable provided each compact portion of $M$ is stable.
If $M \subset \mathbb{R}^N$ is an oriented minimal hypersurface and if $X(x) = \left( \frac{d}{dt} \right)_{t=0} \phi_t(x)$ is a normal vectorfield, we can write $X = u\nu$ where $u : M \to \mathbb{R}$ and $\nu$ is the unit normal vectorfield.

Suppose also that $u \equiv 0$ on $\partial M$. Then

$$\left( \frac{d}{dt} \right)^2_{t=0} \text{area}(\phi_t M) = \frac{1}{2} \int_M (|\nabla u|^2 - |A|^2 u^2) \, dS$$

$$= \frac{1}{2} \int_M (-\Delta u - |A|^2 u)u \, dS.$$

**Proof:** $\left( \frac{d}{dt} \right)^2 \text{area}(\phi_t M) = \int (\frac{d}{dt})^2 J_m(D\phi_t) \, dS = \ldots.$
1. A complete, stable, orientable minimal surface in $\mathbb{R}^3$ is a plane.

2. If $M$ is a stable, orientable minimal surface in $\mathbb{R}^3$, then

$$|A(p)| \ \text{dist}(p, \partial M) \leq C$$

for some $C < \infty$.

As usual, 1 and 2 are equivalent.

Also, a version of 2 holds in Riemannian 3 manifolds.
Lemma

Let $M$ be a complete, simply connected surface with $K \leq 0$. Let $A(r) = A_p(r)$ be the area of the geodesic ball $B_r$ of radius $r$ about some point $p$. Let

$$\theta(M) = \lim_{r \to \infty} \frac{A(r)}{\pi r^2}.$$

Then

$$\theta(M) = 1 - \frac{1}{2\pi} \int_M K \, dS = 1 + \frac{TC(M)}{2\pi}.$$

Corollary

If $M$ (as above) is a minimal surface in $\mathbb{R}^3$ and $\theta(M) < 3$, then $TC(M) < 4\pi$, and therefore $M$ is a plane.
Proof of lemma: Let \( L(r) \) be the length of \( \partial B_r \). Then \( A' = L \), so

\[
A'' = L' = \int_{\partial B_r} k \, ds = 2\pi - \int_{B_r} K \, dS.
\]

Thus

\[
\lim_{r \to \infty} A''(r) = 2\pi - \int_M K \, dS = 2\pi + TC(M).
\]

The result follows easily. \( \square \)
Proof of the stability theorem for complete, simply connected $M$: Suppose $M$ is not a plane. Then by the preceding corollary, $\theta(M) > 3$. Thus $\frac{A(r)}{\pi r^3} > 3$ for large $r$.

But in the problem session you will prove that if $\frac{A(r)}{\pi r^2} > \frac{4}{3}$, then $B_r$ is unstable. (This fact is due to Pogorelov.)
The result for non-simply connected $M$ follows, because for oriented minimal hypersurfaces $M$, stability of $M$ is equivalent to stability of the universal cover. (Fischer-Colbrie/Schoen and Do Carmo/Peng.)