Our goal today: existence and regularity of a surface of least area bounded by a smooth, simple closed curve \( \Gamma \) in \( \mathbb{R}^N \).

As mentioned in lecture 1, the solution depends (in an interesting way!) on what we mean by “surface”, “area”, and “bounded by”. There are different possible definitions of these terms, and they lead to different versions of the Plateau problem.
In the most classical version of the Plateau problem:

- “surface” means “continuous mapping $F : D \rightarrow \mathbb{R}^n$ of a disk into $\mathbb{R}^n$”,
- “bounded by $\Gamma$” means “such that $F : \partial D \rightarrow \Gamma$ is a monotonic parametrization”, and
- “area” means mapping area (as in multivariable calculus):

$$A(F) := \int J(DF) \, dS$$

where

$$J(DF) = \sqrt{|F_x|^2|F_y|^2 - (F_x \cdot F_y)^2}.$$

($F_x = \partial F/\partial y$ and $F_y = \partial F/\partial y$.)
The Douglas-Rado Theorem

**Theorem**

Let $\Gamma$ be a smooth, simple closed curve in $\mathbb{R}^N$. Let $\mathcal{C}$ be the class of continuous maps $F : \overline{D} \to \mathbb{R}^N$ such that $F|D$ is piecewise smooth and such that $F : \partial D \to \Gamma$ is a monotonic parametrization. Then there exists a map $F \in \mathcal{C}$ that minimizes the mapping area $A(F)$.

Indeed, there exists such a map that is harmonic and almost conformal, and is a smooth immersion except (possibly) at isolated points (“branch points”).

**Remark**: We need some condition like piecewise smoothness to guarantee that the integrand in $A(F)$ makes sense. Another possible choice (which works equally well and is better in some ways) is to require that $F$ be locally lipschitz on the interior of the disk. By Rademacher’s Theorem, such an $F$ is differentiable almost everywhere.
The Direct Method for Minimization Problems: Let $\alpha$ be the infimum of $A(F)$ among such $F$.

Then there exists a minimizing sequence $F_i$: a sequence $F_i \in C$ such that $A(F_i) \to \alpha$.

Hope: There exists a convergent subsequence $F_i \to F$, where $A(F) = \alpha$.

For the direct method to work we need:

- A compactness theorem, and
- Lowersemicontinuity: $A(F) \leq \lim \inf A(F_i)$. 
In our setting, a minimizing sequence need not have a convergent subsequence.

For example, there exists a minimizing sequence $F_i$ such that the images $F_i(D)$ converge as sets to all of $\mathbb{R}^N$:

$$\text{dist}(p, F_i(D)) \to 0 \quad \text{for every } p \in \mathbb{R}^N.$$ 

One can also find a minimizing sequence $F_i$ such that $F_i \mid D$ converges pointwise to a constant map.
The remedy: energy instead of area

To avoid such pathologies, instead of using an arbitrary minimizing sequence, we choose a well-behaved minimizing sequence.

For that, we make use of the energy functional.

The energy of a map $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$ is

$$E(F) = \frac{1}{2} \int_M |DF|^2 \, dS$$

where $|DF|^2 = |F_x|^2 + |F_y|^2$. 
We need several facts about energy:

**Lemma (Area-Energy Inequality)**

For $F \in \mathcal{C}$,

$$A(F) \leq E(F),$$

with equality if and only if $F$ is almost conformal.

**Proof**: For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, one easily checks that

$$\sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \leq \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2),$$

with equality if and only if $\mathbf{u}$ are $\mathbf{v}$ orthogonal and have the same length. Apply that fact to $F_x$ and $F_y$ and integrate. \qed
Lemma

Suppose $F : \overline{D} \to \mathbb{R}^N$ is smooth and harmonic. Then

$$E(F) \leq E(G)$$

for all smooth $G : \overline{D} \to \mathbb{R}^N$ with $G|\partial D = F|\partial D$, with equality if and only if $G = F$.

Proof: Let $V = G - F$. Then

$$E(G) = E(F + V) = E(F) + E(V) + \int DF \cdot DV \, dS$$

$$= E(F) + E(V) - \int \Delta F \cdot V \, dS$$

$$= E(F) + E(V).$$

(Note: this holds for domains in arbitrary Riemannian manifolds.)
Before proving the Douglas-Rado Theorem, we need one more lemma:

**Lemma (Smoothing Lemma)**

If $F \in C$ and $\epsilon > 0$, then there is a smooth map $G : \overline{D} \to \mathbb{R}^N$ such that

$$A(G) \leq A(F) + \epsilon.$$  

The lemma is certainly plausible, but the proof is perhaps not so obvious. If time permits, I’ll give the proof later.
Proof of the Douglas-Rado Theorem

**Theorem**

Let $\Gamma$ be a smooth, simple closed curve in $\mathbb{R}^N$. Let $\mathcal{C}$ be the class of continuous maps $F : \overline{D} \to \mathbb{R}^N$ such that $F|D$ is piecewise smooth and such that $F : \partial D \to \Gamma$ is a monotonic parametrization.

Then there exists a map $F \in \mathcal{C}$ that minimizes the mapping area $A(F)$.

Indeed, there exists such a map that is harmonic and almost conformal, and is a smooth immersion except (possibly) at isolated points ("branch points").
Proof: Let $\alpha = \inf \{ A(F) : F \in C \}$.

Claim: If $\epsilon > 0$, then there exists a smooth, harmonic map $H \in C$ with $E(H) \leq \alpha + \epsilon$.

Furthermore, there is such a map such that $F(a) = \hat{a}$, $F(b) = \hat{b}$, and $F(c) = \hat{c}$, where $a$, $b$, and $c$ are three given points on $\partial D$ and $\hat{a}$, $\hat{b}$, and $\hat{c}$ are three given points on $\Gamma$.

Proof of claim: By definition of $\alpha$, there is a map $F \in C$ with $A(F) < \alpha + \epsilon$. By the smoothing lemma, we can choose $F$ to be smooth.

Note that although $F$ is smooth, its image need not be a smooth surface. That is, $F$ need not be an immersion. To get around this, for $\delta > 0$, we define a new map

$$F_\delta : \overline{D} \to \mathbb{R}^n \times \mathbb{R}^2 \cong \mathbb{R}^{n+2},$$

$$F_\delta(z) = (F(z), \delta z).$$

By choosing $\delta$ small, we can assume that $A(F_\delta) < \alpha + \epsilon$. 
Now $F_\delta(\overline{D})$ is a smooth, embedded disk. Hence (by existence of conformal coordinates and the Riemann mapping theorem), we can parametrize $F_\delta(\overline{D})$ by a smooth conformal map $G : \overline{D} \to \mathbb{R}^N$.

In fact, there are many such conformal parametrizations, since if $G$ is one such parametrization and if $u : \overline{D} \to \overline{D}$ is conformal, then $G \circ u$ is also such a parametrization. We choose the conformal parametrization such that

$$G(a), G(b), \text{ and } G(c)$$

project onto $\hat{a}, \hat{b}, \text{ and } \hat{c}$.

Now $A(G) = A(F_\delta)$, so $A(G) < \alpha + \epsilon$.

Also, since $G$ is conformal, its energy and area are equal, so

$$E(G) < \alpha + \epsilon.$$
Now let $G' = \pi \circ G$, where $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is the projection map. Then

$$E(G') \leq E(G) < \alpha + \epsilon.$$ 

Finally, let $H$ be the harmonic map with the same boundary values as $G'$. Then

$$E(H) \leq E(G') < \alpha + \epsilon.$$ 

This completes the proof of the claim. \qed
We now return to the proof of the Douglas-Rado Theorem. By the claim, we can find a sequence of smooth, harmonic maps $H_i \in C$ such that

$$E(H_i) \to \alpha = \inf_{F \in C} A(F).$$

Furthermore, we can choose the $H_i$ so that they map $a$, $b$, and $c$ in $\partial D$ to $\hat{a}$, $\hat{b}$, and $\hat{c}$ in $\Gamma$.

By the maximum principle for harmonic functions, the $H_i$ are uniformly bounded:

$$\max |H_i(\cdot)| \leq \max_{p \in \Gamma} |p|.$$
Equicontinuity claim: The maps $H_i$ are equicontinuous.

This claim is a property of harmonic maps. It only uses following: (i) that $H_i|\partial D$ parametrizes $\Gamma$ monotonically, (ii) the “three point condition”, and (iii) a uniform bound on $E(H_i)$.

I’ll skip the proof for lack of time. (See “Courant-Lebesgue Lemma”.)

By equicontinuity, we can (by passing to a subsequence) assume that the $H_i$ converge uniformly to a limit map $H$.

As already mentioned, $H$ is harmonic on the interior. The uniform convergence implies $H \in C$, so

$$\alpha \leq A(H) \leq E(H) \leq \lim \inf E(H_i) \leq \alpha.$$  

Since $A(H) = E(H)$, the map is almost conformal.
The proof of the Douglas-Rado Theorem only gives continuity (actually Hölder continuity) at the boundary.

However, it was later proved that the map is smooth on the closed disk (Heinz, Hildebrand, Lewy, Kinderlehrer).
Let $F : D \to \mathbb{R}^N$ be a non-constant, harmonic, almost conformal map (such as given by the Douglas-Rado theorem).

Recall that harmonicity of $F$ means that the map $F_z = \frac{1}{2}(F_x - iF_y)$ from $D$ to $\mathbb{C}^n$ is holomorphic. Thus $F_z$ can vanish only at isolated points. Those points are called “branch points”. Away from those points, the map is a smooth, conformal immersion.

Using the Weierstrass Representation, it is easy to give examples of minimal surfaces with branch points. But are there area-minimizing examples?
Theorem (Federer, following Wirtinger)

Let $M$ be a complex variety in $\mathbb{C}^n$. Then (as a real variety in $\mathbb{R}^{2n}$) $M$ is absolutely area minimizing in the following sense: if $S$ is a compact portion of $M$, and if $S'$ is an oriented variety with the same oriented boundary as $S$, then $\text{area}(S) \leq \text{area}(S')$.

The map

$$F : D \subset \mathbb{R}^2 \cong \mathbb{C}^2 \to \mathbb{R}^4 \cong \mathbb{C}^2 \quad F(z) = (z^2, z^3)$$

has a branch point at the origin.

By Federer-Wirtinger theorem, $F$ is area minimizing.
Osserman and Gulliver proved in $\mathbb{R}^3$ (or more generally in any Riemannian 3-manifold) that the Douglas-Rado solution cannot have any interior branch points. Thus (away from the boundary), the map $F$ is a smooth immersion.

One of the longest open questions in minimal surface theory is: can the Douglas-Rado solution have boundary branch points?
There are some situations in which boundary branch points are known not to occur:

1. If $\Gamma$ is extreme, i.e., if it lies on the boundary of a compact, convex region. (In this case, one need not assume area minimizing – minimal suffices.)

2. If $\Gamma$ is real analytic.
Surfaces other than disks

Let $\Gamma$ be a simple closed curve in $\mathbb{R}^n$. Does $\Gamma$ bound a least-area surface of genus 1?

Not necessarily. Consider a planar circle $\Gamma$ in $\mathbb{R}^3$. By the maximum principle, $\Gamma$ bounds only one minimal surface: the flat disk $M$ bounded by $\Gamma$.

In that example, we can take a minimizing sequence of genus 1 surfaces, but in the limit, the handle will shrink to point, and we’ll end up with the disk.
Technically speaking, the planar circle does bound a least area genus 1 surface in the sense of mappings. Let $\Sigma$ be a smooth genus 1 surface consisting of a disk with a handle attached. There is a smooth map $F : \Sigma \rightarrow M$ that collapses the handle to the center $p$ of the disk $M$ bounded by $\Gamma$, and that maps the rest of $\Sigma$ diffeomorphically to $M \setminus \{p\}$.

However, there is no “nice” area minimizing map $F : \Sigma \rightarrow \mathbb{R}^3$ with boundary $\Gamma$. For example, there is no such map that is an immersion except at isolated points.
**Definition**

Let $\Gamma$ be a smooth, simple closed curve in $\mathbb{R}^n$. Let $\alpha(g)$ be the infimum of the area of genus $g$ surfaces bounded by $\Gamma$.

**Proposition**

$\alpha(g + 1) \leq \alpha(g)$.

**Proof**: Take a genus $g$ surface whose area is close to $\alpha(g)$, and then attach a very small handle. \qed
Theorem (Douglas)

If $\alpha(g) < \alpha(g - 1)$, then there exists a domain $\Sigma$ consisting of a genus $g$ Riemann surface with an open disk removed, and a continuous map

$$F : \Sigma \rightarrow \mathbb{R}^n$$

that is harmonic and almost conformal in the interior of $F$ and such that

$$A(F) = \alpha(g).$$

Proof is similar to the proof of the Douglas-Rado Theorem, but more complicated because not all genus $g$ domains are conformally equivalent. For example, up to conformal equivalence, there is a 3-parameter family of genus 1 domains with one boundary component.
The Douglas theorem can be restated (slightly informally) as follows:

**Theorem**

Let $g$ be a nonnegative integer. The least area among all surfaces of genus $\leq g$ bounded by $\Gamma$ is attained by a harmonic, almost conformal map.

**Proof (using the Douglas Theorem):** Let $k$ be the smallest integer such that $\alpha(k) = \alpha(g)$. Then $k \leq g$, and either $k = 0$ or $k > 0$.

If $k = 0$, then the Douglas-Rado solution is a disk that attains the desired infimum $\alpha(g) = \alpha(0)$.

If $k > 0$, then $\alpha(k) < \alpha(k - 1)$, so the genus $k$ surface given by the Douglas Theorem attains the desired infimum $\alpha(g) = \alpha(k)$. 
Summary: For a fixed genus $g$ and curve $\Gamma$, we cannot in general minimize area among surfaces of genus $= g$ and get a nice surface: minimizing sequences may converge to surfaces of lower genus.

However, we can always minimal area among surfaces of genus $\leq g$: the minimum will be attained by a harmonic, almost conformal map.

Intuitively, the Douglas theorem is true because when we take the limit of a minimizing sequence of genus $g$ surfaces, we can lose handles but we can’t gain them.
The Gulliver-Osserman Theorem also holds for these higher genus surfaces: in $\mathbb{R}^3$ (and in 3-manifolds) they must be smooth immersions (except possibly at the boundary)
Let $N$ be a Riemannian 3-manifold and let $F : \overline{D} \to N$ be a least-area disk (parametrized almost conformally) with a smooth boundary curve $\Gamma$. Suppose $F(D)$ is disjoint from $\Gamma$. Then $F$ is a smooth embedding.

The disjointness hypothesis holds in many situations of interest. In particular, it holds if $\Gamma$ lies on the boundary of a compact, convex subset of $\mathbb{R}^3$. (This follows from the strong maximum principle.)

More generally, it holds if $N$ is a compact 3-manifold with mean convex boundary and if $\Gamma$ lies in the $\partial N$. (Mean convexity of the boundary means that the mean curvature vector at each point of the boundary is a nonnegative multiple of the inward unit normal.)
Idea of the proof: Suppose $M$ is immersed but not embedded. One can show that it contains an arc along which it intersects itself transversely. One can cut and paste $M$ along such arcs to get a new piecewise smooth (but not smooth) surface $\tilde{M}$.

This surgery is likely to produce a surface of higher genus.

However, Meeks and Yau show that it is possible to do the surgery in such a way that $\tilde{M}$ is still a disk. Thus

$$\text{area}(\tilde{M}) = \text{area}(M),$$

so $\tilde{M}$ is also area minimizing. However, where $\tilde{M}$ has corners, one can round the corners to get a disk with less area than $\tilde{M}$, a contradiction.