Park City Lectures on Eigenfunctions, Lecture 5: $L^p$ norms of eigenfunctions

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Shapes and sizes of eigenfunctions

This lecture is devoted to quantitative properties of $L^2$ normalized eigenfunctions of compact Riemannian manifolds $(M, g)$. We measure the size of $\varphi_{\lambda}$ by the $L^p$ norms, $||\varphi_{\lambda}||_p$.

The main questions are:

- How large can $||\varphi_{\lambda}||_p$ be, as $(M, g)$ runs over all possible Riemannian manifolds and $\varphi_{\lambda}$ runs over all possible eigenfunctions of $\Delta_g$?
- What types of eigenfunctions have extremal $L^p$ norms? Which $(M, g)$ can have such eigenfunctions?
- How do eigenfunctions concentrate? Which ones are most concentrated? How do eigenfunction concentration reflect the global geometry of $(M, g)$?
- We also broaden our focus to include quasi-modes or approximate eigenfunctions. They are often “semi-classical Lagrangian distributions”. We ask the same questions about them.
**$L^p$ norms**

We recall that

$$\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda, \quad \int_M |\varphi_\lambda|^2 dV = 1.$$

Let $\{\varphi_j(x)\}$ be an orthonormal basis of eigenfunctions. To deal with possible multiplicities of eigenvalues, we consider the eigenspaces

$$V_\lambda = \{\varphi : \Delta \varphi = -\lambda^2 \varphi\},$$

and measure the growth rate of $L^p$ norms by

$$L^p(\lambda, g) = \sup_{\varphi \in V_\lambda : \|\varphi\|_2 = 1} \|\varphi\|_{L^p}. \quad (1)$$

We also consider

$$\ell^p(\lambda, g) = \inf_{\varphi \in V_\lambda : \|\varphi\|_2 = 1} \|\varphi\|_{L^p}. \quad (2)$$
Example: Irrational flat torus

Let \((M, g)\) be a flat irrational torus \(\mathbb{R}^n/L\) where \(L \subset \mathbb{R}^n\) is a co-compact lattice. In this case, the eigenfunctions are \(e^{i \langle \lambda, x \rangle}\) with \(\lambda \in L^*\).

The multiplicities are 2 and it is easy to see that

\[
L^p(\lambda, g) = \ell^p(\lambda, g) = 1.
\]

How often do we find uniformly bounded eigenfunctions? Are there any examples besides the flat torus? (In fact no! if you restrict \((M, g)\) to cases with integrable geodesic flow).
Sogge upper bounds on $L^p$ norms of eigenfunctions

**Theorem**
(Sogge, 1985)

\[
\sup_{\varphi \in V_\lambda} \frac{\|\varphi\|_p}{\|\varphi\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty
\]  

(3)

where

\[
\delta(p) = \begin{cases} 
  n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\
  \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(n+1)}{n-1}.
\end{cases}
\]  

(4)
\[ \sup_{\varphi \in V_\lambda} \frac{\|\varphi\|_p}{\|\varphi\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty \] (5)

where

\[ \delta(p) = \begin{cases} 
2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} = \frac{1}{2} - \frac{2}{p}, & 6 \leq p \leq \infty \\
\frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{1}{4} - \frac{1}{2p}, & 2 \leq p \leq 6.
\end{cases} \] (6)

The critical index is \( p = 6 \), and then \( \delta(p) = \frac{1}{6} \).
Extremals

The upper bounds are sharp in the class of all \((M, g)\) and are saturated on the round sphere:

- For \(p > \frac{2(n+1)}{n-1}\), zonal (rotationally invariant) spherical harmonics saturate the \(L^p\) bounds. Such eigenfunctions also occur on surfaces of revolution.

- For \(L^p\) for \(2 \leq p \leq \frac{2(n+1)}{n-1}\) the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic. Such eigenfunctions also occur on surfaces of revolution.
The left image is a zonal spherical harmonic of degree $N$ on $S^2$: it has high peaks of height $\sqrt{N}$ at the north and south poles. The right image is a Gaussian beam: its height along the equator is $N^{1/4}$ and then it has Gaussian decay transverse to the equator.

The zonal has high $L^p$ norm due to its high peaks on balls of radius $\frac{1}{N}$. The balls are so small that they do not have high $L^p$ norms for small $p$. The Gaussian beams are not as high but they are relatively high over an entire geodesic.
Spherical harmonics of degree $k$

Let $\mathcal{H}_k \subset L^2(S^2)$ denote the space of spherical harmonics of degree $k$. Then:

- $L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. The sum is orthogonal.
- $Sp(\Delta_{S^n}) = \{\lambda_k = k(k + 1)\}$.
- $\dim \mathcal{H}_k = 2k + 1$

The group of rotations around the $x_3$-axis commutes with the Laplacian. We denote its infinitesimal generator by $L_3 = \frac{\partial}{i\partial \theta}$. We then consider the orthonormal basis of joint eigenfunctions:

$$
\begin{cases}
\Delta_{S^2} Y^k_m = k(k + 1) Y^k_m; \\
\frac{\partial}{i\partial \theta} Y^k_m = m Y^k_m.
\end{cases}
$$
Oscillation of a spherical harmonic $Y^l_m$ of low degree
Oscillation of a spherical harmonic $Y_{\ell}^{m}$ of high degree
Spectral projections kernel $= \text{zonal harmonic}$

A key object in the theory of spherical harmonics is the orthogonal projector

$$
\Pi_k : L^2(S^2) \to \mathcal{H}_k.
$$

It has a kernel $\Pi_k(x, y)$ defined by

$$
\Pi_k f(x) = \int_{S^n} \Pi_k(x, y) f(y) dS(y).
$$

If $\{Y_m^k\}$ is an orthonormal basis of $\mathcal{H}_k$ then

$$
\Pi_k(x, y) = \sum_{j=1}^{d_k} Y_m^k(x) \overline{Y_m^k(y)}.
$$
Coherent state $\Phi_k^x$ at $x$

For each $y$, $\Pi_k(x, y) \in \mathcal{H}_k$. We can $L^2$ normalize this function by dividing by the square root of

$$||\Pi_k(\cdot, y)||^2_{L^2} = \int_{S^n} \Pi_k(x, y)\Pi_k(y, x)dS(x) = \Pi_k(x, x).$$

We get

$$\Phi_k^x(y) = \frac{\Pi_k(x, y)}{\sqrt{\Pi_k(x, x)}}.$$
Coherent states extremize $L^\infty$ norms

This is simple and a standard fact about reproducing kernels. The ‘coherent state’ obtained by pinning the spectral projections kernel $\Pi_\lambda(x, y)$ for an eigenspace $V_\lambda$ at one point $y$ and dividing by its $L^2$ norm is always the extremal for pointwise norm at $y$ among eigenfunctions $\varphi_\lambda \in V_\lambda$

$$\varphi_\lambda(x) = \int_M \Pi_\lambda(x, y) \varphi_\lambda(y) dy$$

$$\Rightarrow |\varphi_\lambda(x)| \leq \sqrt{\int_M |\Pi_\lambda(x, y)|^2 dy} = \sqrt{\Pi_\lambda(x, x)}$$

$$= |\Phi^x_\lambda(x)|.$$

In fact, they extremize $L^p$ norms for $p \geq p_n$
Normalized zonal harmonics

On $S^n$, $\Pi_k(x, x) = C_k$ since it is rotationally invariant and $O(n + 1)$ acts transitively on $S^n$. Its integral is $\dim \mathcal{H}_k$, hence, $\Pi_k(x, x) = \frac{1}{\text{Vol}(S^n)} \dim \mathcal{H}_k$. Hence the normalized projection kernel with ‘peak’ at $y_0$ is

$$Y_0^k(x) = \frac{\Pi_k(x, y_0) \sqrt{\text{Vol}(S^n)}}{\sqrt{\dim \mathcal{H}_k}}.$$  

Here, we put $y_0$ equal to the north pole $(0, 0 \cdots, 1)$. The resulting function is called a zonal spherical harmonic since it is invariant under the group $O(n)$ of rotations fixing $y_0$. 
Highest weight spherical harmonics = Gaussian beam

Another important spherical harmonic is $Y_k$, which is the spherical harmonic in $\mathcal{H}_k$ with the largest eigenvalue of $L_3 = \frac{1}{i} \frac{\partial}{\partial \theta}$, or in other words the highest weight.

It is an example of a Gaussian beam along a closed geodesic— a function like $e^{iks} e^{-ky^2}$ in Fermi normal coordinates on the geodesic, with $s$ arc-length and $y$ the coordinate in the normal direction. It is Gaussian in the normal direction.

$Y_k$ is the restriction of the harmonic polynomial $(x_1 + ix_2)^k$ (up to normalization).
Gaussian Beam: maximizes $L^p$ norms for $2 \leq p \leq 6$
Gaussian Beam 2: again, maximizes $L^p$ norms for $2 \leq p \leq 6$
Intuitive principle

The fact that the zonal harmonics and Gaussian beam are extremals for various $L^p$ norms is due the phase space geometry. They concentrate on certain special kinds of invariant sets for the geodesic flow of $S^2$. In general we expect:

- Extremal eigenfunctions should concentrate in phase space along special Lagrangian (or isotropic submanifolds) $\Lambda \subset S^*M$ invariant under the geodesic flow.
- Extremal eigenfunctions should be “semi-classical Lagrangian distributions”, i.e. quantum states corresponding to invariant Lagrangian submanifolds. They should be oscillatory integrals (WKB).
- Their $L^p$ norms should reflect the singularity of the projection $\pi : \Lambda \to M$. 

Lagrangian distributions

Definition: A semi-classical Lagrangian distribution on a manifold $X$ is a sequence $\{u_k(x)\}$ of functions which may be represented as an oscillatory integral,

$$u_k(x) = \hbar_k^{-N/2} \int_{\mathbb{R}^N} e^{i \frac{\phi(x, \theta)}{\hbar_k}} a(x, \theta, \hbar_k) d\theta.$$ 

We assume that $a(x, \theta, \hbar) \sim \sum_{k=0}^{\infty} \hbar^{\mu+k} a_k(x, \theta)$.

It is assumed that $\hbar_k \to 0$ as $k \to \infty$.

Often one thinks of a continuous parameter $\hbar$:

$$u(x, \hbar) = \hbar^{-N/2} \int_{\mathbb{R}^N} e^{i \frac{\phi(x, \theta)}{\hbar}} a(x, \theta, \hbar) d\theta.$$
Lagrangian distributions

The Lagrangian distribution is associated to the following Lagrangian submanifold of $T^*X$: Let

$$C_\varphi = \{(x, \theta) : d_\theta \varphi)(x, \theta) = 0\}.$$ 

Then consider

$$\iota_\varphi(x, \theta) = (x, d_x \varphi(x, \theta) : (x, \theta) \in C_\varphi\}.$$

Stationary phase methods show that $u_k(x)$ concentrates on the Lagrangian submanifold $\Lambda_\varphi$.

The simplest examples are the exponentials $e^{i\langle k, x \rangle}$ which concentrates on the invariant torus $\xi = k$ in $T^*\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n$. 
Spherical harmonic examples

The normalized joint eigenfunctions on the standard sphere are given by

$$Y_m^N(\theta, \varphi) = \sqrt{(2N + 1) \frac{(N - m)!}{(N + m)!}} P_m^N(\cos \varphi) e^{im\theta},$$

where

$$P_m^N(\cos \varphi) = \frac{1}{2\pi} \int_0^{2\pi} (i \sin \varphi \cos \theta + \cos \varphi)^N e^{-im\theta} d\theta$$

are the Legendre polynomials.

To obtain a Lagrangian distribution, we consider a sequence of $Y_m^N$ with $\frac{m}{N} \to C$ for some $C$. I.e. we consider pairs $k(m_0, N_0)$ lying on a ray in the lattice in $\mathbb{Z}^2$ of $(m, N)$ with $|m| \leq N$. 
What is the Lagrangian submanifold?

The Lagrangian submanifold of $T^*S^2$ associated to $Y_m^N$ with $m/N \rightarrow c$ is the torus in $S^*S^2$ defined by

$$p_{\theta}(x, \xi) := \langle \xi, \frac{\partial}{\partial \theta} \rangle = c.$$ 

This is a level set of the Clairaut integral

$$p_{\theta} : S^*S^2 \rightarrow [-1, 1].$$

Examples: (i) The Lagrangian submanifold associated to the zonal spherical harmonic is the “meridian torus” consisting of geodesics from the north to south poles.
(ii) The Gaussian beam is associated to the unit vectors along the equator— a degenerate Lagrangian torus of dimension 1.
Should one expect sequences of eigenfunctions to be Lagrangian distributions?

NO!! It is very rare.

It is true for joint eigenfunctions of commuting operators such as $\frac{\partial}{\partial x_i}$ on $\mathbb{R}^n/\mathbb{Z}^n$; or $\Delta$ and $L_3 = \frac{\partial}{\partial \theta}$ ($x_3$-axis rotations) for $S^2$.

These are quantum integrable Laplacians: $\Delta$ commutes with $n-1$ other independent operators in dimension $n$. E.g $\Box$ for Schwarzchild spacetime.

But these integrable systems are rare. Almost always, eigenfunctions are not at all like Lagrangian distributions. When the geodesic flow is ergodic they behave like “random waves”.
Intensity plot of an ergodic eigenfunction on an (ergodic) Bunimovich stadium: the mass is diffuse
We say that \((M, g)\) has “maximal \(L^p\) eigenfunction growth” if it possesses a sequence \(\varphi_{\lambda_j k}\) of eigenfunctions which achieve the universal growth bounds.

The standard \(S^n\) has maximal eigenfunction growth. The flat torus does not. Nor do hyperbolic manifolds.

How can one characterize \((M, g)\) of “maximal \(L^p\) eigenfunction growth”?
When does \((M, g)\) possess a sequence of eigenfunctions achieving the maximal sup norm bound

\[
\|\varphi_\lambda\|_{L^\infty} \leq C \lambda^{n-1} \frac{1}{2}.
\]
First condition for maximal eigenfunction growth

**Theorem**

*(Sogge-Z)* If there exists a sequence \( \{ \varphi_{\lambda_{j_k}} \} \) of eigenfunctions which achieves (saturates) the bound \( \| \varphi_{\lambda} \|_{L^\infty} \leq \lambda^{(n-1)/2} \) then there must exist a point \( x \in M \) for which the set

\[
\mathcal{L}_x = \{ \xi \in S^*_x M : \exists T : \exp_x T \xi = x \}
\]

of directions of geodesic loops at \( x \) has positive surface measure. Here, \( \exp \) is the exponential map, and the measure \( |\Omega| \) of a set \( \Omega \) is the one induced by the metric \( g_x \) on \( T^*_x M \). For instance, the poles \( x_N, x_S \) of a surface of revolution \((S^2, g)\) satisfy \( |\mathcal{L}_x| = 2\pi \).
Is the loopset condition sharp?

No. There do exist \((M, g)\) which possess ‘self focal points’ such that every geodesic emanating from the point returns to the point at the same time, yet the eigenfunction bounds are not achieved.

Example: a tri-axial ellipsoid (three distinct axes, so not a surface of revolution). The umbilic points are self-focal points. However, the eigenfunctions which maximize the sup-norm only have \(L^\infty\) norms of order \(\lambda^{\frac{n-1}{2}} / \log \lambda\).
Special points and submanifolds

- The zonal spherical harmonics on $S^2$ or a surface of revolution peak at the “poles”. We call a point $p \in (M, g)$ a pole if all geodesics leaving $p$ return to $p$ at some fixed time $T$. These are the only known (to me) examples where the sup-norm bounds are achieved.

- We also call a point $p$ a self-focal point or blow-down point if all geodesics leaving $p$ loop back to $p$ at a common time $T$. They do not have to be closed geodesics. Examples: umbilic points of ellipsoids; foci of ellipses;

- The only known examples of eigenfunctions saturating low $L^p$ bounds are Gaussian beams.
First return time and map

Let \( T_x : S_x^*M \to \mathbb{R}_+ \cup \{\infty\} \) denote the return time function to \( x \),

\[
T_x(\xi) = \begin{cases} 
\inf\{t > 0 : \exp_x t\xi = x\}, & \text{if } \xi \in \mathcal{L}_x; \\
+\infty, & \text{if no such } t \text{ exists.}
\end{cases}
\]

The first return map is define by

\[
\Phi_x := G_x^{T_x} : \mathcal{L}_x \to S_x^* M.
\]

It was first studied by Safarov et al to obtain a two-term local Weyl law when the set of loops at \( x \) has positive measure.
Additional necessary conditions

For simplicity we assume henceforth that \((M, g)\) is real analytic. Then \(\mathcal{L}_x = S^*_x M\) when \(|\mathcal{L}_x| > 0\).

At a self-focal point, with least common return time \(T_x\) we define the first return map

\[
\Phi_x := G^{T_x} : S^*_x M \rightarrow S^*_x M.
\]

The additional necessary condition, at least in the real analytic case, is that \(\Phi_x\) has no invariant \(L^2\) functions.
In the case of round $S^n$, the first return map $\Phi_x = \text{id}$ for all $x$. Thus every direction is recurrent.

In the case of the triaxial ellipsoid satisfy, $|R_x| = 0$ for each $x \in M$ including the umbilic points where $|L_x| = 1$. The reason is that $\Phi_x$ at the umbilic points is a circle map with two fixed points corresponding to the two closed geodesics through $x$. One is stable, one is unstable. Under iterations, $\Phi_x^n(\xi)$ tends to the stable fixed point for any $\xi \in S_x^*M$ except the unstable fixed point. Hence the only recurrent point is the stable fixed point.
Pole of a surface of revolution and umbilic points on an ellipsoid
New result

**Theorem**

(Z, Sogge 2013 +) If \((M, g)\) is real analytic and has maximal eigenfunction growth, then it possesses a self-focal point whose first return map has an invariant function \(f \in L^2(S^*M, \mu_x)\), where \(\mu_x\) is the Euclidean surface measure. Equivalently, it has an invariant measure \(|f|^2 d\mu_x\) which is absolutely continuous with respect to \(d\mu_x\).
Sketch of proof

- Suffices to show that $R(\lambda, x) = o(\lambda^{m-1})$ uniformly in $x \in M$. Weyl remainder $R(\lambda, x)$ can be expressed as an oscillatory integral over $S^*_x M$ à la Duistermaat-Guillemin, Safarov etc.

- There only exist a finite number $M_T$ of self-focal points $p$ with $\Phi_p \neq Id$ and first return time $\leq T$ on $S^*_p M$ in the real analytic case.

- Ergodic theory shows that if $\Phi_p$ has no $L^2$-invariant function, then $R(\lambda, p) = o(\lambda^{m-1})$. Thus, no invariant $L^1$ measures implies small remainders at self-focal points.

- Oscillatory integral analysis shows that $R(\lambda, x) = o(\lambda^{m-1})$ uniformly if $r(x, p) \geq \lambda^{-\frac{1}{2}} \log \lambda$ for all self-focal points $p$.

- Remaining step: $R(\lambda, x) = o(\lambda^{m-1})$ uniformly if $r(x, p) \leq \lambda^{-\frac{1}{2}} \log \lambda$ assuming that $R(\lambda, p) = o(\lambda^{m-1})$. 
Heuristics

We will study the remainder in the pointwise Weyl law rather than eigenfunctions per se. Since eigenfunctions are more intuitive, here is a heuristic guide to what we prove:

- $|\varphi_j(x)|^2$ can only achieve the maximum possible size at or very near a self-focal point.

- If the first return map has no invariant $L^1$ measures at the self-focal point, then $|\varphi_j(x)|^2$ cannot achieve the maximum possible size right at the self-focal point.

- Using the smoothness of the remainder in Weyl’s law, we also prove that if it does not have the maximal order of magnitude at the self-focal point, then it cannot be maximally large very near it either.
Local Weyl law and $L^\infty$ estimates of eigenfunctions

$$E_\lambda(x, x) = \sum_{\lambda_\nu \leq \lambda} |\varphi_\nu(x)|^2$$

$$= (2\pi)^{-n} \int_{p(x, \xi) \leq \lambda} d\xi + R(\lambda, x)$$

with uniform remainder bounds

$$|R(\lambda, x)| \leq C \lambda^{n-1}, \quad x \in M.$$

Since the integral in the local Weyl law is a continuous function of $\lambda$ and since the spectrum of the Laplacian is discrete, this immediately gives

$$\sum_{\lambda_\nu = \lambda} |\varphi_\nu(x)|^2 \leq 2C \lambda^{n-1}$$

which in turn yields

$$L^\infty(\lambda, g) = 0(\lambda^{\frac{n-1}{2}})$$

(9)

on any compact Riemannian manifold.
Local Weyl law with remainder

The eigenfunction bounds are based on the following estimates for the local Weyl remainder.

**Theorem**

Let $R(\lambda, x)$ denote the remainder for the local Weyl law at $x$. Then

$$R(\lambda, x) = o(\lambda^{n-1}) \text{ if } |\mathcal{L}_x| = 0.$$  \hfill (10)

Additionally, if $|\mathcal{L}_x| = 0$ then, given $\varepsilon > 0$, there is a neighborhood $\mathcal{N}$ of $x$ and a $\Lambda = \langle \infty$, both depending on $\varepsilon$ so that

$$|R(\lambda, y)| \leq \varepsilon\lambda^{n-1}, \ y \in \mathcal{N}, \ \lambda \geq \Lambda.$$  \hfill (11)
Notation for smoothed remainder

Let $\hat{\rho} \in C_0^\infty$ be an even function with $\hat{\rho}(0) = 1$, $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, and $\hat{\rho}_T(t) = \hat{\rho}(\frac{t}{T})$.

The classical cosine Tauberian method to determine Weyl asymptotics + remainder is to study

$$\rho_T * dN(\lambda, x) = a_0(x) \lambda^{n-1} + \lambda^{n-1} R(\lambda, x, T),$$

(12)

where $a_0(x)$ is a smooth density ($= a$ constant $C_m$ in our case). We use large $T \iff$ long time behavior of the geodesic flow.
Oscillatory integral formula

By the usual parametrix construction: there exist phases $\tilde{t}_j$ and amplitudes such that

$$\rho_T \ast dN(\lambda, x) = \int_{\mathbb{R}} \hat{\rho}(\frac{t}{T}) e^{i\lambda t} U(t, x, x) dt$$

$$\simeq \lambda^{n-1} \sum_j \int_{S^*_x M} e^{i\lambda \tilde{t}_j(x, \xi)} ((\hat{\rho}_Ta_{j0})) |d\xi| + O(\lambda^{n-2}),$$

where the remainder is uniform in $x$.

Uses long time parametrix for $U(t) = e^{it\sqrt{\Delta}}$, polar coordinates in $T^* M$, and stationary phase in $dt d\tau$. $\tilde{t}_j$ is the value of the phase $\varphi_j(t, x, x, \xi)$ at the critical point.
Many charts, many oscillatory integrals

The index $j$ runs over charts and local oscillatory integrals. Minor technical nuisance: we need to keep track of the number of charts as $T$ grows; we won’t emphasize this again. Then,

$$
\rho_T \ast dN(\lambda, x) = a_0 \lambda^{n-1} + a_1 \lambda^{n-2} + \lambda^{n-1} \sum_{j=1}^{\infty} R_j(\lambda, x, T) + o_T(\lambda^{n-1}),
$$

with uniform remainder in $x$. 
Following Safarov, at a self-focal point $x$ define

$$U_x : L^2(S_x^* M, d\mu_x) \to L^2(S_x^* M, d\mu_x), \quad U_x f(\xi) := f(\Phi_x(\xi)) \sqrt{J_x(\xi)}.$$  \hspace{1cm} (13)

Here, $J_x$ is the Jacobian of the map $\Phi_x$, i.e. $\Phi_x^*|d\xi| = J_x(\xi)|d\xi|$.

Also define

$$U_x^{\pm}(\lambda) = e^{i\lambda T^{\pm}_x} U_x^{\pm}.$$  \hspace{1cm} (14)
If $\hat{\rho} = 0$ in a neighborhood of $t = 0$ then

$$\rho' \ast N(\lambda, x) = \lambda^{n-1} \sum_{k \in \mathbb{Z} \setminus 0} \int_{L_x} \hat{\rho}(kT(\xi))U_x(\lambda)^k \cdot 1d\xi + o_x(\lambda^{n-1}).$$

Here, $L_x = \{\xi \in S^*_x \mathcal{M} : \exp_x \xi = x\}$ is the set of loops at $x$. Recall that

$$U_x f(\xi) = f(\Phi_x(\xi)) \sqrt{J_\Phi},$$

and that $U_x(\lambda)f = e^{i\lambda \tilde{t}(x, \xi)} U_x f$.

In the real analytic case, $L_x = S^*_x \mathcal{M}$ or has measure zero. In the latter case the integral is zero. The remainder is NON-UNIFORM in $x$, so we cannot use this directly and must study its proof.
Decomposition into almost loops and far from loops

Pick $f \geq 0 \in C_0^\infty(\mathbb{R})$ which equals 1 on $|s| \leq 1$ and zero for $|s| \geq 2$ and split up the $j$th term into two terms using $f(\epsilon^{-2}|\nabla_\xi \tilde{t}_j|^2)$ and $1 - f(\epsilon^{-2}|\nabla_\xi \tilde{t}_j|^2)$:

$$R_j(\lambda, x, T) = R_{j1}(\lambda, x, T) + R_{j2}(\lambda, x, T),$$  \hspace{1cm} (15)

where

$$R_{j1}(\lambda, x; \epsilon) := \int_{S_{\hat{x}}^* M} e^{i\lambda \tilde{t}_j} f(\epsilon^{-1}|\nabla_\xi \tilde{t}_j(x, \xi)|^2)(\hat{\rho}(T_x(\xi)))a_0(T_x(\xi), x, \xi)d\xi $$  \hspace{1cm} (16)

The second term $R_{j2}$ comes from the $1 - f(\epsilon^{-2}|\nabla_\xi T_x(\xi)|^2)$ term.

**Lemma**

*For* $\epsilon \geq \lambda^{-\frac{1}{2}} \log \lambda$ *we have*

$$\sup_{x \in M} |R_2(\lambda, x, \epsilon)| \leq C(\log \lambda)^{-1}.$$
Decomposition of $R_1$ into loops and almost loops

On the critical set where $\{\nabla_\xi \tilde{t}_j(x, \xi) = 0\} \subset S_x^* M$, we get

$$\int_{\mathcal{L}_x} e^{i\lambda T_x(\xi)} \hat{\rho}(T_x(\xi)) a_0(T_x, \xi, x\xi) r^*_n |d\xi|.$$ \hspace{1cm} (17)

We may decompose $R_{j1}$ into

$$R_{j1}(x, \epsilon, T) = \int_{\mathcal{L}_x} e^{i\lambda T_x(\xi)} \hat{\rho}(T_x(\xi)) a_0(T_x, \xi, x\xi)|d\xi| + \tilde{R}_{j1}, \hspace{1cm} (18)$$

where

$$\tilde{R}_{j1}(x, \epsilon, T) = \int_{|\nabla_\xi \tilde{t}_j| > 0} e^{i\lambda \tilde{t}_j(x, \xi)} f(\epsilon^{-2} |\nabla_\xi \tilde{t}_j(x, \xi)|^2) a_0(T_x(\xi), x, \xi) d\xi.$$ \hspace{1cm} (19)

The integral over the loopset is the $U_x(\lambda)$ term above.
Recap

So far, for every $x \in M$, we decompose $R(\lambda, x)$ into dynamical term on the loop set. It is zero if $x$ is not self-focal point, and it is the whole integral if it is a self-focal point.

If it is non self-focal, we only get a non-dynamical term $\tilde{R}_{j_1}$ and another non-dynamical term, $R_2$ which is small. We show that we can neglect the points outside of $\bigcup_{j=1}^{M} B_\delta(x_j)$ with $\delta = \lambda^{-1/2} \log \lambda$.

We now consider the centers of the balls: i.e. self-focal points.
Here is the main result showing that $R(\lambda, x)$ is small at the self-focal points if there do not exist invariant $L^2$ functions.

**Proposition**
Assume that $U_x$ has no invariant $L^2$ function for any $x$. Then, for all $\eta > 0$, there exists $T$ independent of $x$ so that for every self-focal point $x$,

$$\frac{1}{T} \left| \int_{\mathcal{L}_x} \left| \sum_{k=1}^{\infty} \hat{\rho} \left( \frac{T_x(k)(\xi)}{T} \right) U_x^k \cdot 1 \right| d\xi \right| \leq \eta. \quad (20)$$
Real analyticity implies the number of self-focal points with twisted $\Phi_x$ is finite

Two key points:

- As mentioned above, in the real analytic case, there can only exist a finite number $M(T)$ of self-focal points with return time $\leq T$ if there are no points with $\Phi_x = Id$. Thus, the ergodic theory is uniform! But it only applies at self-focal points. This is the only place that we really use real analyticity.

- Far from focal points, the measure of “almost critical points” (i.e. loops) is small and we can neglect these.
Uniform $o(\lambda^{n-1})$ outside of small balls around self-focal points

We surround each by a ball $B_\delta(x_j)$ with $\delta$ to be chosen later. In $\mathcal{M} \setminus \bigcup_{j=1}^{M} B_\delta(x_j)$, there do not exist almost any critical directions nor almost critical directions. For sufficiently large $T$ (the first return time) and for sufficiently small $\delta$, the remainders $R(\lambda, x)$ are uniformly small in this set. This just follows from the fact that there are almost no critical points.
Estimate near the self-focal points

This reduces the problem to studying remainder estimates for points of \( \bigcup_{j=1}^{M} B_\delta(x_j) \setminus \{x_j\} \). We may let \( \delta = \lambda^{-\frac{1}{2}} \log \lambda \). We show that the remainder \( R(\lambda, x) \) at these points is bounded (up to constants independent of \((x, \lambda)\)) by the remainder at the center the corresponding ball. The reason is that the

\[
|R(x, \lambda, T) - R(x_j, \lambda, T)| \leq C e^{aT} \text{dist}(x, x_j).
\]

Hence if the distance is \( O(\lambda^{-1/2}) \) we may let \( T = \alpha \log \lambda \) to make this term small.

The ergodic theorem is uniform for a finite set of points, so given \( \epsilon \), we may pick \( T \) (i.e. \( \lambda \)) large enough to make \( |R(x_j, \lambda, T)| \leq \epsilon \).
Summing up

- No eigenfunction $\varphi_j(x)$ can be maximally large at a point $x$ which is $\geq \lambda_j^{-\frac{1}{2}} \log \lambda_j$ away from the self-focal points.

- When there are no invariant measures, it also cannot be large at a self-focal point.

- Using the jump of the remainder, we also see that if it is not large at a self-focal point, it cannot be maximally large very near it either.