Park City Lectures on Eigenfunctions, 2:
$\mathcal{H}^{m-1}(\mathcal{N}_\lambda)$
Joint work with C. Sogge

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Nodal sets of eigenfunctions

Let \((M^n, g)\) be a compact \(C^\infty\) Riemannian manifold of dimension \(n\), let \(\varphi_\lambda\) be an \(L^2\)-normalized eigenfunction of the Laplacian,

\[ \Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda, \]

and let

\[ \mathcal{N}_{\varphi_\lambda} = \{ x : \varphi_\lambda(x) = 0 \} \]

be its nodal hypersurface. The hypersurface volume of the nodal set is denoted

\[ |\mathcal{N}_{\varphi_\lambda}| = \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}). \]

**Problem** Give upper and lower bounds on \(\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda})\).
Yau volume conjecture for $C^\infty$ metrics

**Conjecture**
(S. T. Yau 1978): for any $C^\infty$ metric, there exist constants $C, c > 0$ depending only on $(M, g)$ and not on $\lambda$ such that

$$c\lambda \leq \mathcal{H}^{n-1}(N_{\varphi_\lambda}) \leq C\lambda.$$  \hspace{1cm} (1)

Of course, this pre-supposes that $\mathcal{H}^{n-1}(N_{\varphi_\lambda})$ is well-defined. Below we explain that

$$N_{\varphi_\lambda} = N_{\varphi_\lambda}^{\text{reg}} \bigcup \Sigma_{\varphi_\lambda}$$

where the regular subset is a smooth $n - 1$ hypersurface and the singular set has codimension 1 in $N_{\varphi_\lambda}$ (i.e. $\text{codim} 2$ in $M$).
Some Intuition about nodal sets

- Algebraic geometry: Eigenfunctions of eigenvalue $\lambda^2$ are analogues on $(M^n, g)$ of polynomials of degree $\lambda$. Their nodal sets are analogues of (real) algebraic varieties of this degree. The $\lambda_j \to \infty$ is the high degree limit or high complexity limit.

- The surface volume of a real algebraic hypersurface in $\mathbb{R}^{n+1}$ of degree $d$, defined by one homogeneous polynomial equation

$$\mathcal{H}^{n-1}(\{x \in S^n : P_d(x) = 0\}) \leq C_n d.$$ 

- Thus the analogy predicts the Yau upper bound. For the lower bound one should add the condition that the polynomial is harmonic. Otherwise it need not have real zeros. Harmonic homogeneous polynomials are eigenfunctions of $\Delta$ on $S^{n-1}$. 
How long is the nodal line?
In the real analytic case, Donnelly-Fefferman proved the Yau conjectured bounds.

**Theorem**
*(Donnelly-Fefferman, 1988)* Suppose that \((M, g)\) is real analytic and \(\Delta \phi_\lambda = \lambda^2 \phi_\lambda\). Then

\[
 c_1 \lambda \leq H^{n-1}(Z_{\phi_\lambda}) \leq C_2 \lambda.
\]

In the real anaytic case, one can complexify the problems and use tools of complex analysis, which are much more powerful than \(C^\infty\) methods. We will show that it is possible to obtain results on the distribution of nodal sets, not just their \(H^{n-1}\)-measures.
Singular set of $\mathcal{N}_{\varphi_\lambda}$

The following is proved by Hardt-Simon, C. Bär and in a quantitative form by Hardt-Ostman-Ostenhof.

**Theorem**

The singular set

$$\Sigma_{\varphi_\lambda} \subset \mathcal{N}_{\varphi_\lambda}$$

has Hausdorff dimension $n - 2$ and

$$\mathcal{H}^{n-2}(\Sigma_{\varphi_\lambda}) < \infty.$$

Bär uses the Malgrange preparation theorem to express $\varphi_\lambda$ around a zero $x_0$ as

$$\varphi_\lambda(x) = v(x) \left( x_1^k + \sum_{j=0}^{k-1} u_j(x')x_1^j \right)$$

where $\varphi_\lambda$ vanishes to order $k$ at $x_0$ and $u_j(x')$ vanishes to order $k = j$ at 0. Here, the $x_0 = (0,0)$ in the $(x_1, x')$ coordinates and $v \neq 0$ in some neighborhood of 0.
Bounds until recently

There are special bounds on the length of the nodal line for all $C^\infty$ metrics in dimension 2:

$$c\lambda \leq \mathcal{H}^1(\mathcal{N}_{\varphi_\lambda}) \leq C\lambda^{3/2}.$$  

The lower bound was proved by J. Brüning and the upper bound by Donnelly-Fefferman and R. T. Dong.

In dimensions $\geq 3$ the bounds until recently were

$$C^{-\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq \lambda^{C\lambda}. \quad (2)$$

The upper bound was proved by Hardt-Simon, and the lower bound is proved in the book of FH Lin and Q. Han.
New lower bounds on volumes of nodal hypersurfaces: $C^\infty$ case

**Theorem**

(Colding-Minicozzi, Sogge-Z 2011-12) In dimension $n$ and for any $C^\infty$ metric,

$$\mathcal{H}^{n-1}(\mathcal{N}_{\varphi,\lambda}) \geq C_g \lambda^{3-n}$$

- To date, the upper bound $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi,\lambda}) \leq \lambda^{C_\lambda}$ still stands as the best.

- The C-M and the S-Z proofs are quite different, so it may be surprising that they give the same exponent.
The S-Z proof is based on two ingredients:

- An identity for \((\Delta + \lambda^2)|\varphi_\lambda|\) as a distribution. Note that it equals zero except when \(\varphi_\lambda(x) = 0\). Hence it is supported on the nodal set.

- Use of a “designer reproducing kernel” adapted to \(\varphi_\lambda\) and of an oscillatory integral construction of the kernel. This is the semi-classical analysis part. But it is local and does not use global harmonic analysis.
The identity

The proof is based on the following identity, inspired by a closely related identity of R. T. Dong:

**Proposition**

For any $C^\infty$ Riemannian manifold $(M, g)$ and any eigenfunction $(\Delta + \lambda^2)\varphi_\lambda = 0$,

\[
\lambda^2 \int_M |\varphi_\lambda| dV = 2 \int_{\mathcal{H}^{n-1}} |\nabla \varphi_\lambda| dS.
\]

Here, $dS = d\mathcal{H}^{n-1}$ is the Riemannian surface measure.
A more general identity

**Proposition**

For any $C^\infty$ Riemannian manifold $(M, g)$ and any eigenfunction $(\Delta + \lambda^2)\varphi_\lambda = 0$, and for any $f \in C^2(M)$,

\[
\int_M ((\Delta + \lambda^2)f) |\varphi_\lambda| dV = 2 \int_{N^\varphi_\lambda} f |\nabla \varphi_\lambda| dS. \tag{4}
\]

It is possible that one can improve the lower bound by finding a clever choice of $f$. One can use cutoffs to balls of radius $C\lambda^{-1}$. The identity above also can be used to prove some equidistribution results.
Application to nodal set volumes

The lower bound on nodal volumes is a simple consequence of the identity and the following lemma:

**Lemma**
If $\lambda > 0$ then $\|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}$.

**Proof of lower bound from Lemma:**
\[
\lambda^2 \int_M |\varphi_\lambda| \, dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla_g \varphi_\lambda|_g \, dS \leq 2 |\mathcal{N}_{\varphi_\lambda}| \|\nabla_g \varphi_\lambda\|_{L^\infty(M)}
\]
\[
\lesssim 2 |\mathcal{N}_{\varphi_\lambda}| \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}.
\]

The factor $\|\varphi_\lambda\|_{L^1}$ cancels from the two sides!
Proof of the identity

We have,

\[ M = \bigcup_{j=1}^{N_+(\lambda)} D_j^+ \cup \bigcup_{k=1}^{N_- (\lambda)} D_k^- \cup \mathcal{N}_\lambda, \]

where the \( D_j^+ \) and \( D_k^- \) are the positive and negative nodal domains of \( \phi_\lambda \), i.e., the connected components of the sets \( \{ \phi_\lambda > 0 \} \) and \( \{ \phi_\lambda < 0 \} \).

Let us assume for the moment that 0 is a regular value for \( \phi_\lambda \), i.e., \( \Sigma = \emptyset \). Then each \( D_j^+ \) has smooth boundary \( \partial D_j^+ \), and so if \( \partial \nu \) is the Riemann outward normal derivative on this set, by the Gauss-Green formula we have

\[
\int_{D_j^+} ((\Delta + \lambda^2)f) |\phi_\lambda| dV = \int_{D_j^+} ((\Delta + \lambda^2)f) \phi_\lambda dV
\]

\[ = \int_{D_j^+} f(\Delta + \lambda^2)\phi_\lambda dV - \int_{\partial D_j^+} f \partial _\nu \phi_\lambda dS \]

\[ = \int_{\partial D_j^+} f |\nabla \phi_\lambda| dS, \]
Details of proof

We use that $-\partial_\nu \varphi_\lambda = |\nabla \varphi_\lambda|$ since $\varphi_\lambda = 0$ on $\partial D_j^+$ and $\varphi_\lambda$ decreases as it crosses $\partial D_j^+$ from $D_j^+$. A similar argument shows that

$$\int_{D_k^-} ((\Delta + \lambda^2)f) |\varphi_\lambda| \, dV = -\int_{D_k^-} ((\Delta + \lambda^2)f) \varphi_\lambda \, dV$$

$$= \int f \partial_\nu \varphi_\lambda \, dS = \int_{\partial D_k^-} f |\nabla \varphi_\lambda| \, dS,$$  \hspace{1cm} (6)

using in the last step that $\varphi_\lambda$ increases as it crosses $\partial D_k^-$ from $D_k^-$. 

Summing up

If we sum these two identities over \( j \) and \( k \), we get

\[
\int_{M} ((\Delta + \lambda^2)f) |\varphi_{\lambda}| \, dV = \sum_{j} \int_{D_{j}^+} ((\Delta + \lambda^2)f) |\varphi_{\lambda}| \, dV \\
+ \sum_{k} \int_{D_{k}^-} ((\Delta + \lambda^2)f) |\varphi_{\lambda}| \, dV \\
= \sum_{j} \int_{\partial D_{j}^+} f \, |\nabla \varphi_{\lambda}| \, dS \\
+ \sum_{k} \int_{\partial D_{k}^-} f \, |\nabla \varphi_{\lambda}| \, dS = 2 \int_{\mathcal{N}_{\lambda}} f \, |\nabla \varphi_{\lambda}| \, dS,
\]

using the fact that \( \mathcal{N}_{\lambda} \) is the disjoint union of the \( \partial D_{j}^+ \) and the disjoint union of the \( \partial D_{k}^- \).
If 0 is not a regular value of $\varphi_\lambda$ we use the Gauss-Green formula for domains with rough boundaries. The preceding argument yields

$$\int_{D_j^+} \left( (\Delta + \lambda^2) f \right) |\nabla \varphi_\lambda| \, dV = \int_{\partial D_j^+} f |\nabla \varphi_\lambda| \, dS,$$

and a similar identity for the negative nodal domains. Since $N_\lambda \setminus \Sigma$ is the disjoint union of each of the $\partial D_j^+$ and the $\partial D_k^-$, we conclude that the same equation holds even when 0 is not a regular value of $\varphi_\lambda$. 
Proof of Lemma

Lemma

If \( \lambda > 0 \) then \( \| \nabla g \varphi_\lambda \|_{L^\infty(M)} \lesssim \lambda^{1 + \frac{n-1}{2}} \| \varphi_\lambda \|_{L^1(M)} \).

The main point is to construct a designer reproducing kernel \( K_\lambda \) for \( \varphi_\lambda \):

Let \( \hat{\rho} \in C_0^\infty(\mathbb{R}) \) satisfy \( \rho(0) = \int \hat{\rho} \, dt = 1 \). Define the operator

\[
\rho(\lambda - \sqrt{\Delta}) : L^2(M) \to L^2(M)
\]  

by

\[
\rho(\lambda - \sqrt{\Delta}) f = \int \hat{\rho}(t) e^{it\lambda} e^{-it\sqrt{-\Delta}} f \, dt.
\]

Then (7) is a function of \( \Delta \) and has \( \varphi_\lambda \) as an eigenfunction with eigenvalue \( \rho(\lambda - \lambda) = \rho(0) = 1 \). Hence,

\[
\rho(\lambda - \sqrt{\Delta}) \varphi_\lambda = \varphi_\lambda.
\]
A good choice of $\rho$

We may choose $\rho$ further so that $\hat{\rho}(t) = 0$ for $t \notin [\epsilon/2, \epsilon]$.

**CLAIM** If supp $\hat{\rho} \subset [\epsilon/2, \epsilon]$, then the kernel $K_\lambda(x, y)$ of $\rho(\lambda - \sqrt{\Delta})$ for $\epsilon$ sufficiently small satisfies

$$|\nabla_g K_\lambda(x, y)| \leq C \lambda^{1+\frac{n-1}{2}}. \quad (8)$$

The Claim proves the Lemma, because

$$\nabla_x \varphi_\lambda(x) = \nabla_x \rho(\lambda - \sqrt{\Delta}) \varphi_\lambda(x)$$

$$= \int_M \nabla_x K_\lambda(x, y) \varphi_\lambda(y) dV(y)$$

$$\leq C \sup_{x,y} |\nabla_x K_\lambda(x, y)| \int_M |\varphi_\lambda| dV$$

$$\leq \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1}$$

which implies the lemma.
Proof of Claim

The gradient estimate on $K_\lambda(x, y)$ is based on the following “parametrix” for the designer reproducing kernel:

**Proposition**

$$K_\lambda(x, y) = \lambda^{\frac{n-1}{2}} a_\lambda(x, y) e^{i\lambda r(x, y)}, \quad (9)$$

where $a_\lambda(x, y)$ is bounded with bounded derivatives in $(x, y)$ and where $r(x, y)$ is the Riemannian distance between points.

To obtain the estimate $|\nabla_g K_\lambda(x, y)| \leq C \lambda^{1+\frac{n-1}{2}}$ it suffices to differentiate this expression. The extra power of $\lambda$ comes from the “phase factor” $e^{i\lambda r(x, y)}$. 
Remarks on the reproducing kernel

There are many reproducing kernels if you only want to reproduce one eigenfunction. A very common choice is the spectral projections operator

$$\Pi_{[\lambda, \lambda+1]}(x, y) = \sum_{j: \lambda_j \in [\lambda, \lambda+1]} \varphi_j(x) \varphi_j(y)$$

for the interval $[\lambda, \lambda + 1]$. It reproduces all eigenfunction $\varphi_k$ with $\lambda_k \in [\lambda, \lambda + 1]$. But it is bad for our application because $\Pi_\lambda(x, x) \simeq \lambda^{n-1}$. This follows from the local Weyl law. Also $\sup_{x, y} |\nabla_x \Pi_\lambda(x, y)| \simeq \lambda^n$. 

Proof of Claim

Let \( U(t) = e^{-it\sqrt{\Delta}} \). We may write

\[
\rho(\lambda - \sqrt{\Delta}) = \int_\mathbb{R} \hat{\rho}(t)e^{it\lambda} U(t, x, y) dt.
\]

But also, for small \( t \) and \( x, y \) near the diagonal one may write

\[
U(t, x, y) = \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} At, (x, y, \theta) d\theta.
\]

Thus,

\[
K_\lambda(x, y) = \int_\mathbb{R} \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} e^{it\lambda} \hat{\rho}(t) At, (x, y, \theta) d\theta dt.
\]
We change variables $\theta \rightarrow \lambda \theta$ to obtain

$$K_\lambda(x, y) = \lambda \int \int_0^\infty e^{i\lambda[\theta(r^2(x,y) - t^2) + t]} \hat{\rho}(t)At, (x, y, \lambda \theta) d\theta dt.$$ 

We then apply stationary phase. The phase is

$$\theta(r^2(x, y) - t^2) + t$$

and the critical point equations are

$$r^2(x, y) = t^2, \quad 2t\theta = 1, \quad (t \in (\epsilon, 2\epsilon)).$$

The power of $\theta$ in the amplitude is $\theta^{n-1}$. The change of variables thus puts in $\lambda^{n+1}/2$. But we get $\lambda^{-1}$ from stationary phase with two variables $(t, \theta)$. 
The value of the phase at the critical point is \( e^{it\lambda} = e^{i\lambda r(x,y)} \). The Hessian in \((t, \theta)\) is \(2t\) and it is invertible. Hence,

\[
K_\lambda(x, y) \simeq \lambda \frac{n-1}{2} e^{i\lambda r(x,y)} a(\lambda, x, y),
\]

where

\[
a \sim a_0 + \lambda^{-1} a_{-1} + \cdots
\]

and

\[
a_0 = A_0(r(x, y), (x, y, \frac{2}{r(x, y)})).
\]
Discussion of the result

The lower bound is far away from the $\lambda$ conjectured by Yau, which we know is correct at least in the real analytic case. So the method is missing something. For which $\varphi_\lambda$ do we expect the identity

$$\lambda^2 \int_M |\varphi_\lambda|dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla \varphi_\lambda|dS.$$  \hspace{1cm} (10)

to be relatively useless? Answer: when $\varphi_\lambda$ is a Gaussian beam.
Gaussian beams

Gaussian beams are Gaussian shaped lumps transversal to a closed geodesic which are concentrated on $\lambda^{-\frac{1}{2}}$ tubes $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ around closed geodesics and have height (sup norm) $\lambda^\frac{n-1}{4}$.

Outside of the tube $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ of radius $\lambda^{-\frac{1}{2}}$ around the geodesic, the Gaussian beam and all of its derivatives decay like $e^{-\lambda d^2}$ where $d$ is the distance to the geodesic. The identity only sees $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$. 
Gaussian Beam
Hezari-Sogge bound

Hezari-Sogge manipulated the basic identity, by apply the Schwartz inequality to $\int_{\mathcal{N}} |\nabla \varphi| \, dS$ and then using the test function $f = (1 + \lambda^{-2} |\nabla \varphi|^2 + \varphi^2)$.

**Theorem**
*(Hezari-Sogge, 2011)*

$$\mathcal{H}^{n-1}(\mathcal{N}_\lambda) \geq \lambda \|\varphi_\lambda\|^2_{L_1}.$$ 

Thus, the Yau conjectured $\lambda$ lower bound holds whenever $\|\varphi_\lambda\|_{L^1} \geq C_0 > 0$. In recent work, Sogge observed that one can proved the $\lambda$ lower bound unless BOTH of the following hold:

$$\|\varphi_\lambda\|_{L^\infty} \simeq \lambda^{\frac{n-1}{4}}, \quad \|\varphi_\lambda\|_{L^1} \simeq \lambda^{-\frac{n-1}{4}}.$$ 

Both hold for Gaussian beams!
Gaussian beams are negligible outside of $\lambda^{-\frac{1}{2}}$ tubes $T_{\lambda^{-\frac{1}{2}}}(\gamma)$ around closed geodesics and have height (sup norm) $\lambda^{-\frac{n-1}{4}}$ all along the geodesic.

Hence their $L^1$ norms decrease like $\lambda^{-\frac{(n-1)}{4}}$. In physics language, the exponential decay of Gaussian beams means that the complement of the geodesic is a “forbidden region” for the eigenfunction. The identity does not detect zeros well in forbidden regions.
The same methods apply to any level set $\mathcal{N}_{\varphi_\lambda}^c := \{ \varphi_\lambda = c \}$. Let $\text{sign}(x) = \frac{x}{|x|}$.

**Proposition**

For $c \in \mathbb{R}$

$$
\lambda^2 \int_{\varphi_\lambda \geq c} \varphi_\lambda dV = \int_{\mathcal{N}_{\varphi_\lambda}^c} |\nabla \varphi_\lambda| dS \leq \lambda^2 \text{Vol}(M)^{1/2}.
$$

Consequently, if $c > 0$

$$
\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^c) + \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^{-c}) \geq C_g \lambda^{2 - \frac{n+1}{2}} \int_{|\varphi_\lambda| \geq c} |\varphi_\lambda| dV.
$$