1. Introduction

These lectures are devoted to nodal geometry of eigenfunctions $\varphi_\lambda$ of the Laplacian $\Delta_g$ of a Riemannian manifold $(M^m, g)$ of dimension $m$ and to the associated problems on $L^p$ norms of eigenfunctions. The manifolds are generally assumed to be compact, although the problems can also be posed on non-compact complete Riemannian manifolds. The emphasis of these lectures is on real analytic Riemannian manifolds. We use real analyticity because the study of both nodal geometry and $L^p$ norms simplifies when the eigenfunctions are analytically continued to the complexification of $M$. Although we emphasize the Laplacian, analogous problems and results exist for Schrödinger operators $-\frac{\hbar^2}{2}\Delta_g + V$ for certain potentials $V$. We now state state the main results, some classical and some relatively new, that we concentrate on in these lectures.

The study of eigenfunctions of $\Delta_g$ and $-\frac{\hbar^2}{2}\Delta_g + V$ on Riemannian manifolds is a branch of harmonic analysis. In these lectures, we emphasize high frequency (or semi-classical) asymptotics of eigenfunctions and their relations to the global dynamics of the geodesic flow $G^t : S^*M \to S^*M$ on the unit cosphere bundle of $M$. Here and henceforth we identity vectors and covectors using the metric. As in [Ze3] we give the name “Global Harmonic Analysis” to the use of global wave equation methods to obtain relations between eigenfunction behavior and geodesics. Some of the principal results in semi-classical analysis are purely local and do not exploit this connection. The relations between geodesics and eigenfunctions belongs to the general correspondence principle between classical and quantum mechanics. The correspondence principle has evolved since the origins of quantum mechanics [Sch] into a systematic theory of Semi-Classical Analysis and Fourier integral operators, of which [HoI, HoII, HoIII, HoIV] and [Zw] give systematic presentations; see also §1.13 for further references. Quantum mechanics provides not only the intuition and techniques for the study of eigenfunctions, but in large part also provides the motivation. Readers who are unfamiliar with quantum mechanics are encouraged to read the physics literature. Standard texts are Landau-Lifschitz [LL] and Weinberg [Wei]. The reader might like to see the images recently produced by physicists using quantum microscopes to directly observe nodal sets of excited hydrogen atoms [St].

1.1. The eigenvalue problem on a compact Riemannian manifold. The (negative) Laplacian $\Delta_g$ of $(M^m, g)$ is the unbounded essentially self-adjoint operator on $C^\infty_0(M) \subset L^2(M, dV_g)$ defined by the Dirichlet form

$$D(f) = \int_M |\nabla f|^2 dV_g,$$

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where $\nabla f$ is the metric gradient vector field and $|\nabla f|$ is its length in the metric $g$. Also, $dV_g$ is the volume form of the metric. In terms of the metric Hessian $Dd$,

$$\Delta f = \text{trace } Ddf.$$ 

In local coordinates,

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right),$$

in a standard notation that we assume the reader is familiar with (see e.g. [BGM, Ch BGM, Ch] if not). A more geometric definition uses at each point $p$ an orthonormal basis \{\mathbf{e}_j\}_{j=1}^m of $T_pM$ and geodesics $\gamma_j$ with $\gamma_j(0) = p$, $\gamma'_j(0) = \mathbf{e}_j$. Then

$$\Delta f(p) = \sum_j \frac{d^2}{dt^2} f(\gamma_j(t)).$$

We refer to [BGM] (G.III.12).

**Exercise 1.** Let $m = 2$ and let $\gamma$ be a geodesic arc on $M$. Calculate $\Delta f(s,0)$ in Fermi normal coordinates along $\gamma$.

Background: Define Fermi normal coordinates $(s,y)$ along $\gamma$ by identifying a small ball bundle of the normal bundle $N\gamma$ along $\gamma(s)$ with its image (a tubular neighborhood of $\gamma$) under the normal exponential map, $\exp_{\gamma(s)} y \nu_{\gamma(s)}$. Here, $\nu_{\gamma(s)}$ is the unit normal at $\gamma(s)$ (fix one of the two choices) and $\exp_{\gamma(s)} y \nu_{\gamma(s)}$ is the unit speed geodesic in the direction $\nu_{\gamma(s)}$ of length $y$.

The eigenvalue problem is

$$\Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda, \quad \text{and we assume throughout that } \varphi_\lambda \text{ is } L^2\text{-normalized,}$$

$$||\varphi_\lambda||^2_{L^2} = \int_M |\varphi_\lambda|^2 dV = 1.$$ 

When $M$ is compact, the spectrum of eigenvalues of the Laplacian is discrete there exists an orthonormal basis of eigenfunctions. We fix such a basis \{\varphi_j\} so that

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle_{L^2(M)} := \int_M \varphi_j \varphi_k dV_g = \delta_{jk}$$

If $\partial M \neq \emptyset$ we impose Dirichlet or Neumann boundary conditions. Here $dV_g$ is the volume form. When $M$ is compact, the spectrum of $\Delta_g$ is a discrete set repeated according to multiplicity. Note that $\{\lambda_j\}$ denote the frequencies, i.e. square roots of $\Delta$-eigenvalues. We mainly consider the behavior of eigenfunctions in the 'high frequency' (or high energy) limit $\lambda_j \to \infty$.

The Weyl law asymptotically counts the number of eigenvalues less than $\lambda$,

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = \frac{|B_n|}{(2\pi)^n} \text{Vol}(M,g) \lambda^n + O(\lambda^{n-1}).$$
Here, \(|B_n|\) is the Euclidean volume of the unit ball and \(Vol(M, g)\) is the volume of \(M\) with respect to the metric \(g\). The size of the remainder reflects the measure of closed geodesics \([DG, HoIV]\). It is a basic example of global the effect of the global dynamics on the spectrum. See \(\S\) 1.12 and \(\S\) 6 for related results on eigenfunctions.

(1) In the aperiodic case where the set of closed geodesics has measure zero, the Duistermaat-Guillemin-Ivrii two term Weyl law states

\[ N(\lambda) = \# \{ j : \lambda_j \leq \lambda \} = c_m Vol(M, g) \lambda^m + o(\lambda^{m-1}) \]

where \(m = \dim M\) and where \(c_m\) is a universal constant.

(2) In the periodic case where the geodesic flow is periodic (Zoll manifolds such as the round sphere), the spectrum of \(\sqrt{\Delta}\) is a union of eigenvalue clusters \(C_N\) of the form

\[ C_N = \{ (N + \beta) + \mu_{N_i}, i = 1 \ldots d_N \} \]

with \(\mu_{N_i} = 0(N^{-1})\). The number \(d_N\) of eigenvalues in \(C_N\) is a polynomial of degree \(m - 1\).

Remark: The proof that the spectrum is discrete is based on the study of spectral kernels such as the heat kernel or Green’s function or wave kernel. The standard proof is to show that \(\Delta g^{-1}\) (whose kernel is the Green’s function, defined on the orthogonal complement of the constant functions) is a compact self-adjoint operator. By the spectral theory for such operators, the eigenvalues of \(\Delta g^{-1}\) are discrete, of finite multiplicity, and only accumulate at 0. Although we concentrate on parametrix constructions for the wave kernel, one can construct the Hadamard parametrix for the Green’s function in a similar way. Proofs of the above statements can be found in \([GSj, DSj, Zw, HoIII]\).

1.2. Nodal and critical point sets. The focus of these lectures is on nodal hypersurfaces

(5) \(\mathcal{N}_{\varphi_\lambda} = \{ x \in M : \varphi_\lambda(x) = 0 \}\).

The main problems on nodal sets is to determine the hypersurface volume \(\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda})\) and ideally the distribution of nodal sets. Closely related are the other level sets

(6) \(\mathcal{N}_{\varphi_j}^a = \{ x \in M : \varphi_j(x) = a \}\)

and sublevel sets

(7) \(\{ x \in M : |\varphi_j(x)| \leq a \}\).

The zero level is distinguished since the symmetry \(\varphi_j \to -\varphi_j\) in the equation preserves the nodal set.

Remark: Nodals sets belong to individual eigenfunctions. To the author’s knowledge there do not exist any results on averages of nodal sets over the spectrum in the sense of \((\text{I})-\text{(I0)}\).
That is, we do not know of any asymptotic results concerning the functions
\[ Z_f(\lambda) := \sum_{j: \lambda_j \leq \lambda} \int_{N_{\varphi_j}} f dS, \]
where \( \int_{N_{\varphi_j}} f dS \) denotes the integral of a continuous function \( f \) over the nodal set of \( \varphi_j \).
When the eigenvalues are multiple, the sum \( Z_f \) depends on the choice of orthonormal basis.

Randomizing by taking Gaussian random combinations of eigenfunctions simplifies nodal problems profoundly, and are studied in many articles.

One would also like to know the “number” and distribution of critical points,
\[ \mathcal{C}_{\varphi_j} = \{ x \in M : \nabla \varphi_j(x) = 0 \}. \] (8)
In fact, the critical point set can be a hypersurface in \( M \), so for counting problems it makes more sense to count the number of critical values,
\[ \mathcal{V}_{\varphi_j} = \{ \varphi_j(x) : \nabla \varphi_j(x) = 0 \}. \] (9)
At this time of writing, there exist almost no rigorous upper bounds on the number of critical values, so we do not spend much space on them here.

The frequency \( \lambda \) of an eigenfunction is a measure of its “complexity”, similar to specifying the degree of a polynomial, and the high frequency limit is the large complexity limit. A sequence of eigenfunctions of increasing frequency oscillates more and more rapidly and the problem is to find its “limit shape”. Sequences of eigenfunctions often behave like “Gaussian random waves” but special ones exhibit highly localized oscillation and concentration properties.

1.3. Motivation. Before stating the problems and results, let us motivate the study of eigenfunctions and their high frequency behavior. The eigenvalue problem (1) arises in many areas of physics, for example the theory of vibrating membranes. But renewed motivation to study eigenfunctions comes from quantum mechanics. As is discussed in any textbook on quantum physics or chemistry (see e.g. [LL, Wei]), the Schrödinger equation resolves the problem of how an electron can orbit the nucleus without losing its energy in radiation. The classical Hamiltonian equations of motion of a particle in phase space are orbits of Hamilton’s equations
\[
\begin{align*}
\frac{dx_j}{dt} &= \frac{\partial H}{\partial \xi_j}, \\
\frac{d\xi_j}{dt} &= -\frac{\partial H}{\partial x_j},
\end{align*}
\]
where the Hamiltonian
\[ H(x, \xi) = \frac{1}{2} |\xi|^2 + V(x) : T^*M \to \mathbb{R} \]
is the total Newtonian kinetic + potential energy. The idea of Schrödinger is to model the electron by a wave function \( \varphi_j \) which solves the eigenvalue problem
\[ \hat{H} \varphi_j := (-\frac{\hbar^2}{2} \Delta + V) \varphi_j = E_j(\hbar) \varphi_j, \] (10)
for the Schrödinger operator \( \hat{H} \), where \( V \) is the potential, a multiplication operator on \( L^2(\mathbb{R}^3) \). Here \( \hbar \) is Planck’s constant, a very small constant. The semi-classical limit \( \hbar \to 0 \)
is mathematically equivalent to the high frequency limit when $V = 0$. The time evolution of an 'energy state' is given by

$$U_{\hbar}(t) \varphi_j := e^{-i\frac{t}{\hbar}(-\frac{\hbar^2}{2}\Delta + V)} \varphi_j = e^{-i\frac{E_j(t)}{\hbar}} \varphi_j.$$  \hfill (11)

The unitary operator $U_{\hbar}(t)$ is often called the propagator. In the Riemannian case with $V = 0$, the factors of $\hbar$ can be absorbed in the $t$ variable and it suffices to study

$$U(t) = e^{it\sqrt{\Delta}}.$$  \hfill (12)

An $L^2$-normalized energy state $\varphi_j$ defines a probability amplitude, i.e. its modulus square is a probability measure with

$$|\varphi_j(x)|^2 dx = \text{ the probability density of finding the particle at } x.$$  \hfill (13)

According to the physicists, the observable quantities associated to the energy state are the probability density (13) and 'more generally' the matrix elements

$$\langle A \varphi_j, \varphi_j \rangle = \int \varphi_j(x) A \varphi_j(x) dV$$  \hfill (14)

of observables ($A$ is a self adjoint operator, and in these lectures it is assumed to be a pseudo-differential operator). Under the time evolution (11), the factors of $e^{-i\frac{E_j(t)}{\hbar}}$ cancel and so the particle evolves as if "stationary", i.e. observations of the particle are independent of the time $t$.

Modeling energy states by eigenfunctions resolves the paradox of particles which are simultaneously in motion and are stationary, but at the cost of replacing the classical model of particles following the trajectories of Hamilton’s equations by ‘linear algebra’, i.e. evolution by (11). The quantum picture is difficult to visualize or understand intuitively. Moreover, it is difficult to relate the classical picture of orbits with the quantum picture of eigenfunctions.

The study of nodal sets was historically motivated in part by the desire to visualize energy states by finding the points where the quantum particle is least likely to be. In fact, just recently (at this time or writing) the nodal sets of the hydrogen atom energy states have become visible to microscopes [St].

1.4. Nodal hypersurface volumes for $C^\infty$ metrics. Let us proceed to the rigorous results whose proofs we will sketch in these lectures. In the late 70’s, S. T. Yau conjectured that for general $C^\infty (M, g)$ of any dimension $m$ there exist $c_1, C_2$ depending only on $g$ so that

$$\lambda \lesssim H^{m-1}(N_{\varphi_\lambda}) \lesssim \lambda.$$  \hfill (15)

Here and below $\lesssim$ means that there exists a constant $C$ independent of $\lambda$ for which the inequality holds. The upper bound of (15) is the analogue for eigenfunctions of the fact that the hypersurface volume of a real algebraic variety is bounded above by its degree. The lower bound is specific to eigenfunctions. It is a strong version of the statement that 0 is not an “exceptional value” of $\varphi_\lambda$. Indeed, a basic result is the following classical result, apparently due to R. Courant (see [Br])). It is used to obtain lower bounds on volumes of nodal sets:

**Proposition 1.** For any $(M, g)$ there exists a constant $A > 0$ so that every ball of $(M, g)$ of radius greater than $\frac{A}{\lambda}$ contains a nodal point of any eigenfunction $\varphi_\lambda$. 

We sketch the proof in §2.2 for completeness, but leave some of the proof as an exercise to the reader.

The lower bound of (15) was proved for all $C^\infty$ metrics for surfaces, i.e. for $m = 2$ by Brüning [Br]. For general $C^\infty$ metrics in dimensions $\geq 3$, the known upper and lower bounds are far from the conjecture (15). At present the best lower bound available for general $C^\infty$ metrics of all dimensions is the following estimate of Colding-Minicozzi [CM]; a somewhat weaker bound was proved by Sogge-Zelditch [SoZ] and the later simplification of the proof [SoZa] turned out to give the same bound as [CM]. We sketch the proof from [SoZa].

**Theorem 2.**

$$\lambda^{1 - \frac{n-1}{2}} \lesssim \mathcal{H}^{m-1}(\mathcal{N}_\lambda),$$

The original result of [SoZa] is based on lower bounds on the $L^1$ norm of eigenfunctions. Further work of Hezari-Sogge [HS] shows that the Yau lower bound is correct when one has $||\varphi_\lambda||_{L^1} \geq C_0$ for some $C_0 > 0$. It is not known for which $(M, g)$ such an estimate is valid. At the present time, such lower bounds are obtained from upper bounds on the $L^1$ norm of $\varphi_j$. The study of $L^p$ norms of eigenfunctions is of independent interest and we discuss some recent results which are not directly related to nodal sets in §9 and in §12. The study of $L^p$ norms splits into two very different cases: there exists a critical index $p_n$ depending on the dimension of $M$, and for $p \geq p_n$ the $L^p$ norms of eigenfunctions are closely related to the structure of geodesic loops (see §12). For $2 \leq p \leq p_n$ the $L^p$ norms are governed by different geodesic properties of $(M, g)$ which we discuss in §12.

We also review an interesting upper bound due to R. T. Dong and Donnelly-Fefferman in dimension 2, since the techniques of proof of [Dong] seem capable of further development.

**Theorem 3.** For $C^\infty (M, g)$ of dimension 2,

$$\mathcal{H}^1(\mathcal{N}_\lambda) \lesssim \lambda^{3/2}.$$  

1.5. **Nodal hypersurface volumes for real analytic** $(M, g)$. In 1988, Donnelly-Fefferman [DF] proved the conjectured bounds for real analytic Riemannian manifolds (possibly with boundary). We re-state the result as the following

**Theorem 4.** Let $(M, g)$ be a compact real analytic Riemannian manifold, with or without boundary. Then

$$c_1 \lambda \lesssim \mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \lesssim \lambda.$$  

We sketch the proof of the upper bound in §10.

1.6. **Dynamics of the geodesic or billiard flow.** There are two broad classes of results on nodal sets and other properties of eigenfunctions:

- Local results which are valid for any local solution of (11), and which often use local arguments. For instance the proof of Proposition 1 is local.

- Global results which use that eigenfunctions are global solutions of (11), or that they satisfy boundary conditions when $\partial M \neq \emptyset$. Thus, they are also satisfy the unitary evolution equation (11). For instance the relation between closed geodesics and the remainder term of Weyl’s law is global (11)-(10).
Global results often exploit the relation between classical and quantum mechanics, i.e. the relation between the eigenvalue problem \((E, U)\) and the geodesic flow. Thus the results often depend on the dynamical properties of the geodesic flow. The relations between eigenfunctions and the Hamiltonian flow are best established in two extreme cases: (i) where the Hamiltonian flow is completely integrable on an energy surface, or (ii) where it is ergodic. The extremes are illustrated below in the case of (i) billiards on rotationally invariant annulus, (ii) chaotic billiards on a cardioid.

A random trajectory in the case of ergodic billiards is uniformly distributed, while all trajectories are quasi-periodic in the integrable case.

We do not have the space to review the dynamics of geodesic flows or other Hamiltonian flows. We refer to [HK] for background in dynamics and to [Ze, Ze3, Zw] for relations between dynamics of geodesic flows and eigenfunctions.

We use the following basic construction: given a measure preserving map (or flow) \(\Phi : (X, \mu) \to (T, \mu)\) one can consider the translation operator

\[
U_{\Phi} f(x) = \Phi^* f(x) = f(\Phi(x)),
\]

sometimes called the Koopman operator or Perron-Frobenius operator (cf. [RS, HK]). It is a unitary operator on \(L^2(X, \mu)\) and hence its spectrum lies on the unit circle. \(\Phi\) is ergodic if and only if \(U_{\Phi}\) has the eigenvalue 1 with multiplicity 1, corresponding to the constant functions.

The geodesic (or billiard) flow is the Hamiltonian flow on \(T^*M\) generated by the metric norm Hamiltonian or its square,

\[
H(x, \xi) = |\xi|^2_g = \sum_{i,j} g^{ij} \xi_i \xi_j.
\]

In PDE one most often uses the \(\sqrt{H}\) which is homogeneous of degree 1. The geodesic flow is ergodic when the Hamiltonian flow \(\Phi^t\) is ergodic on the level set \(S^*M = \{H = 1\}\).

1.7. Complexification of \(M\) and Grauert tubes. The results of Donnelly-Fefferman Theorem [4] in the real analytic case uses in part the analytic continuation of the eigenfunctions to the complexification of \(M\). One of the themes of these lectures is that nodal problems in the complex domain are simpler than in the real domain.

A real analytic manifold \(M\) always possesses a unique complexification \(M_C\) generalizing the complexification of \(\mathbb{R}^m\) as \(\mathbb{C}^m\). The complexification is an open complex manifold in which \(M\) embeds \(\iota : M \to M_C\) as a totally real submanifold (Bruhat-Whitney).

The Riemannian metric determines a special kind of distance function on \(M_C\) known as a Grauert tube function. In fact, it is observed in [GS1] that the Grauert tube function
is obtained from the distance function by setting $\sqrt{\rho}(\zeta) = i \sqrt{r^2(\zeta, \zeta)}$ where $r^2(x, y)$ is the squared distance function in a neighborhood of the diagonal in $M \times M$.

One defines the Grauert tubes $M_\tau = \{ \zeta \in M_C : \sqrt{\rho}(\zeta) \leq \tau \}$. There exists a maximal $\tau_0$ for which $\sqrt{\rho}$ is well defined, known as the Grauert tube radius. For $\tau \leq \tau_0$, $M_\tau$ is a strictly pseudo-convex domain in $M_C$. Since $(M, g)$ is real analytic, the exponential map $\exp_x t \xi$ admits an analytic continuation in $t$ and the imaginary time exponential map

$$E : B^*_\epsilon M \to M_C, \quad E(x, \xi) = \exp_x i \xi$$

is, for small enough $\epsilon$, a diffeomorphism from the ball bundle $B^*_\epsilon M$ of radius $\epsilon$ in $T^*M$ to the Grauert tube $M_\epsilon$ in $M_C$. We have $E^* \omega = \omega_{T^*M}$ where $\omega = i \partial \bar{\partial} \rho$ and where $\omega_{T^*M}$ is the canonical symplectic form; and also $E^* \sqrt{\rho} = |\xi|$ [GSI, LS1]. It follows that $E^*$ conjugates the geodesic flow on $B^*_\epsilon M$ to the Hamiltonian flow $\exp t H_{\sqrt{\rho}}$ with respect to $\omega$, i.e.

$$E(g^t(x, \xi)) = \exp t \Xi_{\sqrt{\rho}}(\exp_x i \xi).$$

In general $E$ only extends as a diffeomorphism to a certain maximal radius $\epsilon_{\text{max}}$. We assume throughout that $\epsilon < \epsilon_{\text{max}}$.

### 1.8. Equidistribution of nodal sets in the complex domain.

One may also consider the complex nodal sets

$$N_{\varphi_j}^C = \{ \zeta \in M : \varphi_j^C(\zeta) = 0 \},$$

and the complex critical point sets

$$C_{\varphi_j}^C = \{ \zeta \in M : \partial \varphi_j^C(\zeta) = 0 \}.$$

The following is proved in [Ze5]:

**Theorem 5.** Assume $(M, g)$ is real analytic and that the geodesic flow of $(M, g)$ is ergodic. Then for all but a sparse subsequence of $\lambda_j$,

$$\frac{1}{\lambda_j} \int_{N_{\varphi_j}^C} f \omega_{g}^{n-1} \to i \int_{M} f \partial \bar{\partial} \sqrt{\rho} \wedge \omega_{g}^{n-1} \quad \mu(S^n M) \leq 4 \pi r(\varphi_j^C, \varphi_j^C).$$

The proof is based on quantum ergodicity of analytic continuation of eigenfunctions to Grauert tubes and the growth estimates ergodic eigenfunctions satisfy.

We will say that a sequence $\{ \varphi_j \}$ of $L^2$-normalized eigenfunctions is quantum ergodic if

$$\langle A \varphi_j, \varphi_j \rangle \to \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu, \quad \forall A \in \Psi^0(M).$$

Here, $\Psi^s(M)$ denotes the space of pseudodifferential operators of order $s$, and $d\mu$ denotes Liouville measure on the unit cosphere bundle $S^*M$ of $(M, g)$. More generally, we denote by $d\mu_r$ (the surface) Liouville measure on $\partial B^*_r M$, defined by

$$d\mu_r = \frac{\omega^n}{d|\xi|_g} \text{ on } \partial B^*_r M.$$

We also denote by $\alpha$ the canonical action 1-form of $T^*M$. 

}\exp \frac{t}{\tau_0}M \to \tau_0 M_C, \quad E(x, \xi) = \exp_x i \xi
1.9. **Intersection of nodal sets and real analytic curves on surfaces.** To understand the relation between real and complex zeros, we intersect nodal lines and real analytic on surfaces \( \dim M = 2 \). In recent work, intersections of nodal sets and curves have been used in a variety of articles to obtain upper and lower bounds on nodal points and domains. The work often is based on restriction theorems for eigenfunctions. Some of the recent articles on restriction theorems and/or nodal intersections are [TZ, TZ2, GRS, JJ, JJ2, Mar, Yo, Po].

First we consider a basic upper bound on the number of intersection points:

**Theorem 6.** Let \( \Omega \subset \mathbb{R}^2 \) be a piecewise analytic domain and let \( n_{\partial \Omega}(\lambda_j) \) be the number of components of the nodal set of the \( j \)-th Neumann or Dirichlet eigenfunction which intersect \( \partial \Omega \). Then there exists \( C_\Omega \) such that \( n_{\partial \Omega}(\lambda_j) \leq C_\Omega \lambda_j \).

In the Dirichlet case, we delete the boundary when considering components of the nodal set.

The method of proof of Theorem 6 generalizes from \( \partial \Omega \) to a rather large class of real analytic curves \( C \subset \Omega \), even when \( \partial \Omega \) is not real analytic. Let us call a real analytic curve \( C \) a *good* curve if there exists a constant \( a > 0 \) so that for all \( \lambda \) sufficiently large,

\[
(24) \quad \frac{\|\varphi_{\lambda_j}\|_{L^2(\partial \Omega)}}{\|\varphi_{\lambda_j}\|_{L^2(C)}} \leq e^{a \lambda_j}.
\]

Here, the \( L^2 \) norms refer to the restrictions of the eigenfunction to \( C \) and to \( \partial \Omega \). The following result deals with the case where \( C \subset \partial \Omega \) is an *interior* real-analytic curve. The real curve \( C \) may then be holomorphically continued to a complex curve \( C_C \subset \mathbb{C}^2 \) obtained by analytically continuing a real analytic parametrization of \( C \).

**Theorem 7.** Suppose that \( \Omega \subset \mathbb{R}^2 \) is a \( C^\infty \) plane domain, and let \( C \subset \Omega \) be a good interior real analytic curve in the sense of (24). Let \( n(\lambda_j, C) = \#Z_{\varphi_{\lambda_j}} \cap C \) be the number of intersection points of the nodal set of the \( j \)-th Neumann (or Dirichlet) eigenfunction with \( C \). Then there exists \( A_{C, \Omega} > 0 \) depending only on \( C, \Omega \) such that \( n(\lambda_j, C) \leq A_{C, \Omega} \lambda_j \).

The proof of Theorem 7 is somewhat simpler than that of Theorem 6, i.e. good interior analytic curves are somewhat simpler than the boundary itself. On the other hand, it is clear that the boundary is good and hard to prove that other curves are good. A recent paper of J. Jung shows that many natural curves in the hyperbolic plane are ‘good’ [JJ]. See also [ElHajT] for general results on good curves.

The upper bounds are proved by analytically continuing the restricted eigenfunction to the analytic continuation of the curve. We then give a similar upper bound on complex zeros. Since real zeros are also complex zeros, we then get an upper bound on complex zeros. An obvious question is whether the order of magnitude estimate is sharp. Simple examples in the unit disc show that there are no non-trivial lower bounds on numbers of intersection points. But when the dynamics is ergodic we can prove an equi-distribution theorem for nodal intersection points. Ergodicity once again implies that eigenfunctions oscillate as much as possible and therefore saturate bounds on zeros.

Let \( \gamma \subset M^2 \) be a generic geodesic arc on a real analytic Riemannian surface. For small \( \epsilon \), the parametrization of \( \gamma \) may be analytically continued to a strip,

\[
\gamma_C : S_\tau := \{ t + i \tau \in \mathbb{C} : |\tau| \leq \epsilon \} \to M_\tau.
\]
Then the eigenfunction restricted to $\gamma$ is
\[ \gamma^* \varphi^C_j(t + i\tau) = \varphi_j(\gamma C(t + i\tau)) \text{ on } S_\tau. \]

Let
\[ (25) \quad N_{\lambda_j}^\gamma := \{ (t + i\tau : \gamma^* H^C_{\lambda_j}(t + i\tau) = 0) \} \]
be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points.

Then as a current of integration,
\[ (26) \quad [N_{\lambda_j}^\gamma] = i\partial_\bar{\tau} \log \left| \left| \gamma^* \varphi^C_{\lambda_j} (t + i\tau) \right| \right|^2. \]

The following result is proved in \textit{Ze6}:

**Theorem 8.** Let $(M, g)$ be real analytic with ergodic geodesic flow. Then for generic $\gamma$ there exists a subsequence of eigenvalues $\lambda_{jk}$ of density one such that
\[ \frac{i}{\pi \lambda_{jk}} \partial_\bar{\tau} \log \left| \left| \gamma^* \varphi^C_{\lambda_{jk}} (t + i\tau) \right| \right|^2 \to \delta_{\tau=0} ds. \]

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain and are distributed by arc-length measure on the real geodesic.

The key point is that
\[ \frac{1}{\lambda_{jk}} \log |\varphi^C_{\lambda_{jk}}(t + i\tau)|^2 \to |\tau|. \]

Thus, the maximal growth occurs along individual (generic) geodesics.

1.10. **Quantum integrable eigenfunctions.** So far, all of the exact asymptotic results we have discussed assume ergodicity of the geodesic flow. We now give a result in the opposite dynamical extreme where the geodesic flow is completely integrable. Thus, the phase space orbits wind around on invariant Lagrangian tori of dimension $m = \dim M$ rather than (almost surely) winding densely around in $S^*_M$ of dimension $2m - 1$. We in fact need to assume integrability on the quantum level. We only discuss the real analytic case here.

The Laplacian $\Delta$ of a real analytic $(M, g)$ is \textit{quantum completely integrable} or QCI if there exist $m = \dim M$ first-order analytic pseudo-differential operators $P_1, \ldots, P_m$ such that
\[ (27) \quad P_1 = \sqrt{\Delta}, \quad [P_i, P_j] = 0 \]
and whose symbols $(p_1, \ldots, p_m)$ satisfy the non-degeneracy condition $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_m \neq 0$ on a dense open set $\Omega \subset T^*M - 0$. We are assuming that $P_1 = \sqrt{\Delta}$ but it is often simpler to assume that $\sqrt{\Delta}$ is some other function $\hat{H}(P_1, \ldots, P_m)$. Note that the symbols must Poisson commute, \{p_i, p_j\} = 0, i.e. the associated geodesic flow is completely integrable in the classical sense that they generate a Hamiltonian $\mathbb{R}^m$ action. Simple examples of QCI Laplacians in dimension two include flat tori, surfaces of revolution, ellipsoids, and Liouville tori. If one works with Schrödinger operators, then there are many further examples such as the Hydrogen atom, harmonic oscillator, Calogero-Moser Hamiltonian etc.

We denote by $\{\varphi_\alpha\}$ an orthonormal basis of joint eigenfunctions,
\[ (28) \quad P_j \varphi_\alpha = \alpha_j \varphi_\alpha, \quad \langle \varphi_\alpha, \varphi_\alpha' \rangle = \delta_{\alpha, \alpha'}. \]
of the $P_j$ and the joint spectrum of $(P_1, \ldots, P_m)$ by

$$\text{Spec}((P_1, \ldots, P_m) = \Sigma := \{ \vec{\alpha} := (\alpha_1, \ldots, \alpha_m) \} \subset \mathbb{R}^m.$$ 

The eigenvalues of $\sqrt{\Delta}$ are thus of the form $H(\vec{\mu})$ with $\vec{\mu} \in \Sigma$ and the multiplicity of an eigenvalue is the number of $\vec{\mu}$ with a given value of $H(\vec{\mu})$. We refer to the special joint eigenfunctions (28) as the QCI eigenfunctions. The QI eigenfunctions are complex-valued and we consider the nodal sets

$$\text{Re} \varphi_\alpha = 0, \quad \text{Im} \varphi_\alpha = 0$$

of their real or imaginary parts.

A completely integrable system is a non-degenerate Hamiltonian $\mathbb{R}^m$ action on the cotangent bundle $T^*M$ of a manifold. The vector of classical symbols of the $P_j$ (27) defines the moment map

$$\mathcal{P} = (p_1, \ldots, p_m) : T^*M - 0 \to B \subset \mathbb{R}^m$$

of the Hamiltonian action. We assume that the $P_j$ are first order pseudo-differential operators, so that the $p_j$ are homogeneous of degree one and thus the image $B$ is a cone. The level sets $\mathcal{P}^{-1}(b)$ of the moment map consist of a finite union of orbits Hamiltonian flow

$$\Phi_{\vec{t}}(x, \xi) := \exp(t_1 \Xi_{p_1}) \circ \cdots \circ \exp(t_m \Xi_{p_m})(x, \xi), \quad \vec{t} = (t_1, \ldots, t_m),$$

where $\Xi_p$ denotes the Hamiltonian vector field of $p$. When compact, the orbits are tori of dimensions $\leq m$. always exist singular levels.

We say that the Hamiltonian system is toric integrable if the Hamiltonian $\mathbb{R}^n$ action (31) reduces to a Hamilton $\mathbb{T}^m$ action where $\mathbb{T}^m = S^1 \times \cdots \times S^1$ is the $m$-torus. Equivalently, if the integrable system admits global action-angle variables $I_1, \ldots, I_m, \theta_1, \ldots, \theta_m$. By an action variable is meant a Hamiltonian generating a $2\pi$-periodic Hamiltonian flow. We denote the moment map by

$$\mathcal{I} = (I_1, \ldots, I_m) : T^*M - 0 \to B \subset \mathbb{R}^m.$$ 

We assumed above that $p_1 = |\xi|$ but with the generators $I_j$ this is not usually the case; rather there exists a homogeneous function $H$ of degree one so that

$$|\xi| = H(I_1, \ldots, I_m).$$

The level sets $\mathcal{I}^{-1}(b)$ then consist of a single orbit. On the singular levels, the orbit drops dimension or equivalent has an isotropy subgroup of positive dimension, much like points on the divisor at infinity of a toric Kähler manifold.

**Exercise 2.** The geodesic flow of an ellipsoid is not toric integrable. Nor is the geodesic flow of a “peanut of revolution”. Find a geometric argument that proves that the peanut cannot be toric integrable. Hint: what kinds of closed geodesics can occur in the toric integrable case?

Toric integrable systems are always toric on the quantum level in the sense that one can choose generators $\hat{I_1}, \ldots, \hat{I_m}$ of the algebra of pseudo-differential operators commuting with $\Delta$ whose exponentials generate a unitary representation of $\mathbb{T}^m$ on $L^2(M)$, at least up to scalars. That is, the joint spectrum is contained in an off-set of a conic subset $\Lambda$ of a lattice,

$$Sp(\hat{I_1}, \ldots, \hat{I_m}) = \Lambda + \nu \subset \mathbb{Z}^m + \nu.$$
where \( \nu \in (\mathbb{Z}/4)^m \) is a Maslov index. For instance in the case of the standard \( S^2 \) one can choose generators whose spectrum is the set \( \{(m, n + \frac{1}{2}) : -n \leq m \leq n, n \geq 0\} \).

Semiclassical limits are taken along ladders in the joint spectrum. In the case of quantum torus actions, we define rational ladders by

\[
L_\alpha = \mathbb{R} \alpha + \nu, \quad (\alpha \in \Lambda).
\]

Thus, rational rays consist of multiples of a given lattice point.

We refer to a ladder as a regular ladder if \( P^{-1}(\alpha) \) is a regular level, and as a singular ladder if \( P^{-1}(\alpha) \) is a singular level. For simplicity, we only consider limit distribution along ladders for regular levels. We transfer the moment map \( \mathcal{I} \) to \( M_\epsilon \) by

\[
\mathcal{I}_\epsilon : M_\epsilon \to \mathbb{R}^n, \quad \mathcal{I}_\epsilon = \mathcal{I} \circ E^{-1}.
\]

The eigenfunctions \( \varphi_\alpha \) admit holomorphic extensions \( \varphi_\alpha^C \) to a certain Grauert tube \( M_\epsilon \) independent of \( \alpha \). We note that \( \varphi_\alpha \) is normalized to have \( L^2 \)-norm equal to 1 but is only defined up to a unit complex number. Since it is not unique, we consider \( \varphi_\alpha^C(z) \varphi_\alpha^C(y) \) where \( y \) is fixed and \( z \) varies. We then consider the analytic continuations of the real and imaginary parts,

\[
\left( \text{Re} \varphi_\alpha(x) \varphi_\alpha(y) \right)^C, \quad \left( \text{Im} \varphi_\alpha(x) \varphi_\alpha(y) \right)^C.
\]

In the case of a QCI system, \( \varphi_\alpha(x) = \varphi_{-\alpha}(x) \) for \( x \in M \), hence

\[
\text{Re} \left( \text{Re} \varphi_\alpha(\cdot) \varphi_\alpha(y) \right)^C(z) = \frac{1}{2} (\varphi_\alpha^C(z) \varphi_{-\alpha}^C(y) + \varphi_{-\alpha}^C(z) \varphi_\alpha^C(y)).
\]

To illustrate the notation, in the case of \( \mathbb{R}^n/\mathbb{Z}^n \) we have \( \varphi_\alpha(x) = e^{i \langle \alpha, x \rangle} \), \( \text{Re} \varphi_\alpha(x) \varphi_\alpha(y) = \cos \langle \alpha, x - y \rangle \) and \( \left( \text{Re} \varphi_\alpha(\cdot) \varphi_\alpha(y) \right)^C(z) = \cos \langle \alpha, z - y \rangle \). In this example it is natural to set \( y = 0 \).

As discussed above, the key problem in finding the limit distribution of nodal sets in the complex domain is to determine the exponential growth rate of the complexified eigenfunctions. In the QCI case, the growth rates of (36) depend on the ladder \( L_\alpha \). We therefore define

\[
\sqrt{\rho}_\alpha(z) := \lim_{k \to \infty} \frac{1}{k H(\alpha)} \log |\varphi_{k\alpha}^C(z)|
\]

\[
u_{\alpha}(z, y) := \lim_{k \to \infty} \frac{1}{k H(\alpha)} \log |\varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y) + \varphi_{-k\alpha}^C(z) \varphi_{-k\alpha}^C(y)|.
\]

The zero set of \( \sqrt{\rho}_\alpha \) is a real hypersurface in \( M_\epsilon \) known as the Anti-Stokes hypersurface (see §7.)

To determine the exponential growth asymptotics of the sequence \( \{\frac{\varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y)}{|\varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y)|}\}_{k=1}^\infty \), we find a convenient complex oscillatory integral expression for it and then use the method of complex stationary phase. Since the ladder \( L_\alpha \) is fixed, there is a distinguished level set of the moment map in each \( \partial M_\epsilon \).

\[\text{Definition 9. We put}\]

\[\bullet \quad \Lambda_\alpha := \mathcal{I}^{-1}(\frac{\alpha}{H(\alpha)}) \subset T^* M \setminus 0.\]

\[\bullet \quad \Lambda_\alpha^* = E(\Lambda_\alpha) = \mathcal{I}_\epsilon^{-1}(\epsilon \frac{\alpha}{H(\alpha)}) \subset \partial M_\epsilon.\]
These level sets are torus orbits and we view them as the classically allowed set (see §8.1 for background). The main point is that there exists a real \( \tau \in \mathbb{T}^m \) so that \( \Phi^\tau_\alpha y = z \) in this case, and it is then straightforward to determine the asymptotics of \( \{ \frac{e^{k\alpha}(z)}{||\varphi_k||^2} \}_{k=1}^\infty \) when \( z, y \in \Lambda_\alpha \).

The main complication is that for \( z \notin \Lambda_\alpha \), i.e. in the classically forbidden region, there does not exist a critical point of the complex oscillatory integral on the contour of integration. We therefore must locate the dominant critical point in the complexification \( \mathbb{T}^m \times (M_\epsilon)_{\mathbb{C}} \) of the contour, where \((M_\epsilon)_{\mathbb{C}} \) is the complexification of the Grauert tube (viewed as a real manifold). We further must analytically continue the torus action to this complexification. It turns out to be important to distinguish between the analytic continuation of orbits to complex time on \( M_\epsilon \) and the analytic continuation of the group action to \((M_\epsilon)_{\mathbb{C}} \). To explain this and to state the results, we need to introduce some further notation.

**Definition 10.** We denote by \( \Gamma_z : \mathbb{T}^m \to M_\epsilon \) the orbit \( \Gamma_z(t) \) of a point \( z \in M_\epsilon \) under \( \mathbb{T}^m \). We further denote by \( D_z \subset \mathbb{T}^m_{\mathbb{C}} \) the maximal domain of analytic continuation of \( \Gamma_z \),

\[
\Gamma_z : D_z \to M_\epsilon.
\]

Given two regular points \((y, z) \in \partial M_\epsilon^{\text{reg}} \times \partial M_\epsilon^{\text{reg}} \) there exists a unique \( ((\tilde{t} + i\tilde{\tau})(z, y) \in \mathbb{T}^m_{\mathbb{C}} \) so that \( \Gamma_y(\tilde{t} + i\tilde{\tau}) = z \).

**Definition 11.** We define two complex travel times with respect to the complexified torus action:

- Given \( z \in \partial M_\epsilon \) there exists a unique imaginary time \( \tilde{\tau}(z, \alpha) \in \mathbb{R}^m \) so that
  \[
  \Gamma_z(\exp i\tilde{\tau}(z, \alpha)) \in \Lambda_\alpha \subset \partial M_\epsilon.
  \]

- Given a second point \( y \in \partial M_\epsilon \), satisfying \( \mathcal{I}_\epsilon(y) = \frac{\alpha}{H(\alpha)} \), there exists a unique complex time \( \tilde{t}(z, \alpha, y) + i\tau(z, \alpha, y) \) so that
  \[
  \Gamma_z(\exp \tilde{t}(z, y) + i\tau(z, y)) = y.
  \]

The imaginary time \( \tilde{\tau}(z, \alpha) \) is the travel time from \( z \) to \( \Lambda_\alpha \). Note that if we move a point on \( \Lambda_\alpha \), it changes the travel time by a real vector and does not change the imaginary part. On the other hand, \( \tilde{t}(z, y) + i\tau(z, y) \) is the complex travel time from \( z \) to \( y \).

**Theorem 12.** Let \((M, g) \) be a real analytic Riemannian manifold with quantum torus integrable Laplacian, and let \( \{ \varphi_{k\alpha} \} \) be a regular ladder of \( L^2 \)-normalized joint eigenfunctions. Let \( z \in \partial M_\epsilon \) and let \( y \in \mathcal{I}_\epsilon^{-1}(\alpha) \). Then,

\[
\begin{align*}
\sqrt{p}_\alpha(z) &= \frac{1}{2} \langle \frac{\alpha}{H(\alpha)}, \tilde{\tau}(z, \alpha) \rangle + \sqrt{p}(z) \\
u_\alpha(z) &= \max\{ \frac{1}{2} \langle \tilde{\tau}(z, \alpha, y), \frac{\alpha}{H(\alpha)} \rangle + \sqrt{p}(z), -\frac{1}{2} \langle \tilde{\tau}(z, -\alpha, y), \frac{\alpha}{H(\alpha)} \rangle + \sqrt{p}(z) \}
\end{align*}
\]

We note that \( \sqrt{p} \) is the maximal exponent of growth of eigenfunctions, so we must have \( \sqrt{p}_\alpha(z) \leq \sqrt{p}(z) \). Note that \( \sqrt{p}_\alpha = -\sqrt{-p} \) and \( u_\alpha = |\sqrt{p}_\alpha| \).

Combining with Proposition 8.1 gives our main result,
**Corollary** 13. Let \( \mathcal{N}_\alpha^c \) be the complex nodal set of \( \text{Re} \varphi_\alpha \). Then for a regular ladder \( L_\alpha = \{ k\alpha, k \in \mathbb{Z}_+ \} \), \( u_\alpha \) is well-defined and the limit distribution of the nodal set currents along the ladder is given by

\[
\lim_{k \to \infty} \frac{1}{k|\alpha|} \int_{\mathcal{N}_\alpha^c} f \omega_{\rho}^{m-1} \to \frac{i}{\pi} \int_{M_c} f \omega_{\rho}^{m-1} \wedge \frac{i}{\pi} \partial \bar{\partial} \left( \frac{\langle \tau(z, \alpha, y), \alpha \rangle}{H(\alpha)} + \sqrt{\rho(z)} \right).
\]

The restriction to regular ladders is not just technical. The methods and results are different for singular levels, as illustrated by highest weight spherical harmonics (§3.6). They have unusual growth and decay behavior in both the real and complex domain. Eigenfunctions associated to singular levels are important and we plan to study them in a future article. Note that \( \varphi_\alpha^c \) has no complex zeros if and only if \( \langle \tau(z, \alpha) \rangle \) is a harmonic function.

### 1.11. Example: Flat torus.
Flat tori \( \mathbb{R}^n/L \) are quantum integrable with \( \hat{I}_j = \frac{\partial}{\partial x_j} \). The QI joint eigenfunctions are of course the exponentials \( e^{i(x)} \) where \( \bar{\lambda} \in L^* \), the dual lattice to \( L \). The corresponding real eigenfunctions are \( \sin(x, \bar{\lambda}) \), \( \cos(x, \bar{\lambda}) \), and their analytic continuations are \( \sin(\zeta, \bar{\lambda}), \cos(\zeta, \bar{\lambda}) \). All of their complex zeros \( \zeta = x + i\xi \mod L \) are real and satisfy

\[
(38) \quad \sin(x + i\xi, \bar{\lambda}) = 0 \iff \langle x, \bar{\lambda} \rangle \in \pi \mathbb{Z}, \quad \langle \xi, \bar{\lambda} \rangle = 0.
\]

In the case of \( \mathbb{R}^n/Z^n \), the classically allowed region for \( e^{i(x)} \) is the entire torus. Upon analytic continuation we see that along a ray of lattice points, \( e^{i(k\alpha, x + i\xi)} \) is exponentially growing when \( \langle \xi, \alpha \rangle < 0 \) and is exponentially decaying where \( \langle \xi, \alpha \rangle > 0 \). Thus the real hypersurface \( \{ x + i\xi \in (\mathbb{C}^n/L) : \langle \xi, \alpha \rangle = 0 \} \) is the boundary between the two regimes and is thus the anti-Stokes surface

\[
\mathcal{A}_{\mathcal{S}_\alpha} = \{ z : \langle \text{Im} z, \alpha \rangle = 0 \}.
\]

The level set of the moment map is the Lagrangian torus \( \xi = \frac{\xi}{|\alpha|} \); where \( \varphi_\alpha^c \) attains its maximum growth rate. With \( y = \epsilon \frac{\alpha}{|\alpha|} \),

\[
\tau(z, \alpha) = \text{Im} z - \epsilon \frac{\alpha}{|\alpha|}, \quad \tau(z, \alpha, \epsilon \frac{\alpha}{|\alpha|}) = \text{Im} z - \epsilon \frac{\alpha}{|\alpha|},
\]

so that

\[
\begin{cases}
\sqrt{\rho}_\alpha(z) = \langle \text{Im} z - \epsilon \frac{\alpha}{|\alpha|}, \frac{\alpha}{|\alpha|} \rangle + \epsilon = \langle \text{Im} z, \frac{\alpha}{|\alpha|} \rangle,

u_\alpha(z) = \max\{ \langle \text{Im} z - \epsilon \frac{\alpha}{|\alpha|}, \frac{\alpha}{|\alpha|} \rangle + 2\epsilon, \langle \text{Im} z + \epsilon \frac{\alpha}{|\alpha|}, \frac{\alpha}{|\alpha|} \rangle + 2\epsilon \} = |\langle \text{Im} z, \frac{\alpha}{|\alpha|} \rangle|.
\end{cases}
\]

The nodal set of the complexified real part is given by

\[
\cos\langle k\alpha, x + i\xi \rangle = 0 \iff \langle \xi, \alpha \rangle = 0, \quad k\langle \alpha, x \rangle = \pi/2 \mod 2\pi,
\]

so that the limit nodal set lies in \( \mathcal{A}_{\mathcal{S}_\alpha} \) and is uniformly distributed on it.

The explicitly computable examples like the flat torus or sphere are not representative since the torus acts by holomorphic maps in these cases. The holomorphic maps necessarily extend to the zero section and must be lifts of isometries of the base.
1.12. \textit{$L^p$ norms of eigenfunctions.} In §LBintro we mentioned that lower bounds on $H^{n-1}(N_{\varphi_\lambda})$ are related to lower bounds on $||\varphi_\lambda||_{L^1}$ and to upper bounds on $||\varphi_\lambda||_{L^p}$ for certain $p$. Such $L^p$ bounds are interesting for all $p$ and depend on the shapes of the eigenfunctions.

In (WL4) we stated the Weyl law on the number of eigenvalues. There also exists a pointwise local Weyl law which is relevant to the pointwise behavior of eigenfunctions. The pointwise spectral function along the diagonal is defined by

\begin{equation}
E(\lambda, x, x) = N(\lambda, x) := \sum_{\lambda_j \leq \lambda} |\varphi_j(x)|^2.
\end{equation}

The pointwise Weyl law asserts that

\begin{equation}
N(\lambda, x) = \frac{1}{(2\pi)^n} |B^n| \lambda^n + R(\lambda, x),
\end{equation}

where $R(\lambda, x) = O(\lambda^{n-1})$ uniformly in $x$. These results are proved by studying the cosine transform

\begin{equation}
E(t, x, x) = \sum_{\lambda_j \leq \lambda} \cos t\lambda_j |\varphi_j(x)|^2,
\end{equation}

which is the fundamental (even) solution of the wave equation restricted to the diagonal. Background on the wave equation is given in §WAVEAPP12.

We note that the Weyl asymptote $\frac{1}{(2\pi)^n} |B^n| \lambda^n$ is continuous, while the spectral function (SPECFUN39) is piecewise constant with jumps at the eigenvalues $\lambda_j$. Hence the remainder must jump at an eigenvalue $\lambda$, i.e.

\begin{equation}
R(\lambda, x) - R(\lambda - 0, x) = \sum_{j: \lambda_j = \lambda} |\varphi_j(x)|^2 = O(\lambda^{n-1}).
\end{equation}

on any compact Riemannian manifold. It follows immediately that

\begin{equation}
\sup_M |\varphi_j| \lesssim \lambda_j^{\frac{n-1}{2}}.
\end{equation}

There exist $(M, g)$ for which this estimate is sharp, such as the standard spheres. However, it is very rarely sharp and the actual size of the sup-norms and other $L^p$ norms of eigenfunctions is another interesting problem in global harmonic analysis. In [SoZ] it is proved that if the bound (SPECFUN43) is achieved by some sequence of eigenfunctions, then there must exist a “partial blow-down point” or self-focal point $p$ where a positive measure of directions $\omega \in S^*_p M$ so that the geodesic with initial value $(p, \omega)$ returns to $p$ at some time $T(p, \omega)$. Recently the authors have improved the result in the real analytic case, and we sketch the new result in §Lp9.

To state it, we need some further notation and terminology. We only consider real analytic metrics for the sake of simplicity. We call a point $p$ a \textit{self-focal point} or \textit{ablow-down point} if there exists a time $T(p)$ so that $\exp_p T(p) \omega = p$ for all $\omega \in S^*_p M$. Such a point is self-conjugate in a very strong sense. In terms of symplectic geometry, the flowout manifold \begin{equation}
\Lambda_p = \bigcup_{0 \leq t \leq T(p)} G^t S^*_p M
\end{equation}
is an embedded Lagrangian submanifold of $S^*M$ whose projection

$$
\pi : \Lambda_p \to M
$$

has a “blow-down singularity” at $t = 0, t = T(p)$ (see [STZ]). Focal points come in two basic kinds, depending on the first return map

$$
\Phi_p : S_p^* M \to S_p^* M, \quad \Phi_p(\xi) := \gamma'_{p,\xi}(T(p)),
$$

where $\gamma_{p,\xi}$ is the geodesic defined by the initial data $(p, \xi) \in S^*_p M$. We say that $p$ is a pole if

$$
\Phi_p = Id : S_p^* M \to S_p^* M.
$$

On the other hand, it is possible that $\Phi_p = Id$ only on a codimension one set in $S_p^* M$. We call such a $\Phi_p$ twisted.

Examples of poles are the poles of a surface of revolution (in which case all geodesic loops at $x_0$ are smoothly closed). Examples of self-focal points with fully twisted return map are the four umbilic points of two-dimensional tri-axial ellipsoids, from which all geodesics loop back at time $2\pi$ but are almost never smoothly closed. The only smoothly closed directions are the geodesic (and its time reversal) defined by the middle length ‘equator’.

At a self-focal point we have a kind of analogue of \(\Phi_{phl}\) but not on $S^* M$ but just on $S_p^* M$. We define the Perron-Frobenius operator at a self-focal point by

$$
U_x : L^2(S^*_x M, d\mu_x) \to L^2(S^*_x M, d\mu_x), \quad U_x f(\xi) := f(\Phi_x(\xi)) \sqrt{J_x(\xi)}.
$$

Here, $J_x$ is the Jacobian of the map $\Phi_x$, i.e. $\Phi_x^*|d\xi| = J_x(\xi)|d\xi|$.

The new result of C.D. Sogge and the author is the following:

**Theorem 14.** If $(M, g)$ is real analytic and has maximal eigenfunction growth, then it possesses a self-focal point whose first return map $\Phi_p$ has an invariant $L^2$ function in $L^2(S_p^* M)$. Equivalently, it has an $L^1$ invariant measure in the class of the Euclidean volume density $\mu_p$ on $S_p^* M$.

For instance, the twisted first return map at an umbilic point of an ellipsoid has no such finite invariant measure. Rather it has two fixed points, one of which is a source and one a sink, and the only finite invariant measures are delta-functions at the fixed points. It also has an infinite invariant measure on the complement of the fixed points, similar to $\frac{dx}{x}$ on $\mathbb{R}_+$.

The results of [SoZ, STZ, SoZ2] are stated for the $L^\infty$ norm but the same results are true for $L^p$ norms above a critical index $p_m$ depending on the dimension ($\S10$). The analogous problem for lower $L^p$ norms is of equal interest, but the geometry of the extremals changes from analogues of zonal harmonics to analogoues of Gaussian beams or highest weight harmonics. For the lower $L^p$ norms there are also several new developments which are discussed in \S10.

1.13. **Format of these lectures and references to the literature.** In keeping with the format of the Park City summer school, various details of the proof are given as Exercises for the reader. The “details” are intended to be stimulating and fundamental, rather than the tedious and routine aspects of proofs often left to readers in textbooks. As a result, the exercises vary widely in difficulty and amount of background assumed. Problems labelled Problems are not exercises; they are problems whose solutions are not currently known.
The technical backbone of the semi-classical analysis of eigenfunctions consists of wave equation methods combined with the machinery of Fourier integral operators and Pseudo-differential operators. We do not have time to review this theory. The main results we need are the construction of parametrices for the ‘propagator’ \( E(t) = \cos t\sqrt{\Delta} \) and the Poisson kernel \( \exp(-\tau \sqrt{\Delta}) \). We also need Fourier analysis to construct approximate spectral projections \( \rho(\sqrt{\Delta} - \lambda) \) and to prove Tauberian theorems relating smooth expansions and cutoffs.

The books \([GS_1, DS, D_2, GS_2, GST_1, Sogb, Sogb_2, Zw]\) give textbook treatments of the semi-classical methods with applications to spectral asymptotics. Somewhat more classical background on the wave equation with many explicit formulae in model cases can be found in \([TI, TII]\). General spectral theory and the relevant functional analysis can also be found in \([RS]\). The series \([Hol, Hol_II, Hol_III, Hol_IV]\) gives a systematic presentation of Fourier integral operator theory: stationary phase and Tauberian theorems can be found in \([Hol]\), Weyl’s law and spectral asymptotics can be found in \([Hol_III, Hol_IV]\).

In \([Ze_0]\) the author gives a more systematic presentation of results on nodal sets, \( L^p \) norms and other aspects of eigenfunctions. Earlier surveys \([Ze, Ze_2, Ze_3]\) survey related material. Other monographs on \( \Delta \)-eigenfunctions can be found in \([HL]\) and \([Sogb_2]\). The methods of \([HL]\) mainly involve the local harmonic analysis of eigenfunctions and rely more on classical elliptic estimates, on frequency functions and of one-variable complex analysis. The exposition in \([Sogb_2]\) is close to the one given here but does not extend to the recent results that we highlight in these lectures and in \([Ze_0]\).

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## 2. Foundational results on nodal sets

The nodal domains of an eigenfunction are the connected components of \( M \setminus \mathcal{N}_{\varphi,\lambda} \). In the case of a domain with boundary and Dirichlet boundary conditions, the nodal set is defined by taking the closure of the zero set in \( M \setminus \partial M \).

The eigenfunction is either positive or negative in each nodal domain and changes sign as the nodal set is crossed from one domain to an adjacent domain. Thus the set of nodal domains can be given the structure of a bi-partite graph \([HI]\). Since the eigenfunction has one sign in each nodal domain, it is the ground state eigenfunction with Dirichlet boundary conditions in each nodal domain.

In the case of domains \( \Omega \subset \mathbb{R}^n \) (with the Euclidean metric), the Faber-Krahn inequality states that the lowest eigenvalue (ground state eigenvalue, bass note) \( \lambda_1(\Omega) \) for the Dirichlet problem has the lower bound,

\[
\lambda_1(\Omega) \geq |\Omega|^{-\frac{2}{n}} C_n^2 j_{n-2},
\]

where \(|\Omega|\) is the Euclidean volume of \( \Omega \), \( C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \) is the volume of the unit ball in \( \mathbb{R}^n \) and where \( j_{m,1} \) is the first positive zero of the Bessel function \( J_m \). That is, among all domains of a fixed volume the unit ball has the lowest bass note.
2.1. Vanishing order and scaling near zeros. By the vanishing order $\nu(u,a)$ of $u$ at $a$ is meant the largest positive integer such that $D^\alpha u(a) = 0$ for all $|\alpha| \leq \nu$. A unique continuation theorem shows that the vanishing order of an eigenfunction at each zero is finite. The following estimate is a quantitative version of this fact.

**Theorem 2.1.** (see [DF]; [Lin] Proposition 1.2 and Corollary 1.4; and [H] Theorem 2.1.8.) Suppose that $M$ is compact and of dimension $n$. Then there exist constants $C(n), C_2(n)$ depending only on the dimension such that the vanishing order $\nu(u,a)$ of $u$ at $a \in M$ satisfies $\nu(u,a) \leq C(n) N(0,1) + C_2(n)$ for all $a \in B_{1/4}(0)$. In the case of a global eigenfunction, $\nu(\varphi, a) \leq C(M,g)\lambda$.

Highest weight spherical harmonics $C_n(x_1+ix_2)^N$ on $S^2$ are examples which vanish at the maximal order of vanishing at the poles $x_1 = x_2 = 0, x_3 = \pm 1$.

The following Bers scaling rule extracts the leading term in the Taylor expansion of the eigenfunction around a zero:

**Bers, HW2** Assume that $\varphi_\lambda$ vanishes to order $k$ at $x_0$. Let $\varphi_\lambda(x) = \varphi^{x_0}_k(x) + \varphi^{x_0}_{k+1} + \cdots$ denote the $C^\infty$ Taylor expansion of $\varphi_\lambda$ into homogeneous terms in normal coordinates $x$ centered at $x_0$. Then $\varphi^{x_0}_k(x)$ is a Euclidean harmonic homogeneous polynomial of degree $k$.

To prove this, one substitutes the homogeneous expansion into the equation $\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda$ and rescales $x \to \lambda x$, i.e. one applies the dilation operator

$$D^{x_0}_\lambda \varphi_\lambda(u) = \varphi(x_0 + \frac{u}{\lambda}).$$

The rescaled eigenfunction is an eigenfunction of the locally rescaled Laplacian

$$\Delta^{x_0}_\lambda := \lambda^{-2} D^{x_0}_\lambda \Delta g(D^{x_0}_\lambda)^{-1} = \sum_{j=1}^n \frac{\partial^2}{\partial u_j^2} + \cdots$$

in Riemannian normal coordinates $u$ at $x_0$ but now with eigenvalue 1,

$$D^{x_0}_\lambda \Delta g(D^{x_0}_\lambda)^{-1} \varphi(x_0 + \frac{u}{\lambda}) = \lambda^2 \varphi(x_0 + \frac{u}{\lambda})$$

$$\implies \Delta^{x_0}_\lambda \varphi(x_0 + \frac{u}{\lambda}) = \varphi(x_0 + \frac{u}{\lambda}).$$

Since $\varphi(x_0 + \frac{u}{\lambda})$ is, modulo lower order terms, an eigenfunction of a standard flat Laplacian on $\mathbb{R}^n$, it behaves near a zero as a sum of homogeneous Euclidean harmonic polynomials.

The Bers scaling is used by S.Y. Cheng (see also earlier results of Hartman-Wintner [HW, Ch1, Ch2]) to prove that at a singular point of $\varphi_\lambda$ in dimension two, the nodal line branches in $k$ curves at $x_0$ with equal angles between the curves. For further applications, see [Bes].

**Question** Is there any useful scaling behavior of $\varphi_\lambda$ around its critical points?

2.2. Proof of Proposition 1.8. The proofs are based on rescaling the eigenvalue problem in small balls.

**Proof.** Fix $x_0, r$ and consider $B(x_0, r)$. If $\varphi_\lambda$ has no zeros in $B(x_0, r)$, then $B(x_0, r) \subset D_{j,\lambda}$ must be contained in the interior of a nodal domain $D_{j,\lambda}$ of $\varphi_\lambda$. Now $\lambda^2 = \lambda_1^2(D_{j,\lambda})$ where $\lambda_1^2(D_{j,\lambda})$ is the smallest Dirichlet eigenvalue for the nodal domain. By domain monotonicity of the lowest Dirichlet eigenvalue (i.e. $\lambda_1(\Omega)$ decreases as $\Omega$ increases), $\lambda^2 \leq \lambda_1^2(D_{j,\lambda}) \leq$
\(\lambda_1^2(B(x_0, r))\). To complete the proof we show that \(\lambda_1^2(B(x_0, r)) \leq \frac{C}{r^2}\) where \(C\) depends only on the metric. This is proved by comparing \(\lambda_1^2(B(x_0, r))\) for the metric \(g\) with the lowest Dirichlet Eigenvalue \(\lambda_1^2(B(x_0, cr); g_0)\) for the Euclidean ball \(B(x_0, cr; g_0)\) centered at \(x_0\) of radius \(cr\) with Euclidean metric \(g_0\) equal to \(g\) with coefficients frozen at \(x_0\); \(c\) is chosen so that \(B(x_0, cr; g_0) \subset B(x_0, r, g)\). Again by domain monotonicity, \(\lambda_1^2(B(x_0, r, g)) \leq \lambda_1^2(B(x_0, cr; g))\) for \(c < 1\). By comparing Rayleigh quotients \(\int_{B} |f|^2 dv_g\) one easily sees that \(\lambda_1^2(B(x_0, cr; g)) \leq C\lambda_1^2(B(x_0, cr; g_0))\) for some \(C\) depending only on the metric. But by explicit calculation with Bessel functions, \(\lambda_1^2(B(x_0, cr; g_0)) \leq \frac{C}{r^2}\). Thus, \(\lambda_1^2 \leq \frac{C}{r^2}\).

\[\square\]

For background we refer to [Ch].

2.3. \textbf{A second proof.} Another proof is given in [HL]: Let \(u_r\) denote the ground state Dirichlet eigenfunction for \(B(x_0, r)\). Then \(u_r > 0\) on the interior of \(B(x_0, r)\). If \(B(x_0, r) \subset D_{j, \lambda}\) then also \(\varphi_\lambda > 0\) in \(B(x_0, r)\). Hence the ratio \(\frac{u_r}{\varphi_\lambda}\) is smooth and non-negative, vanishes only on \(\partial B(x_0, r)\), and must have its maximum at a point \(y\) in the interior of \(B(x_0, r)\). At this point (recalling that our \(\Delta\) is minus the sum of squares),

\[\nabla \left( \frac{u_r}{\varphi_\lambda} \right)(y) = 0, \quad -\Delta \left( \frac{u_r}{\varphi_\lambda} \right)(y) \leq 0,\]

so at \(y\),

\[0 \geq -\Delta \left( \frac{u_r}{\varphi_\lambda} \right) = -\frac{\varphi_\lambda \Delta u_r - u_r \Delta \varphi_\lambda}{\varphi_\lambda^2} = -\frac{\lambda_1^2(B(x_0, r)) - \lambda^2}{\varphi_\lambda^2} \varphi_\lambda u_r.\]

Since \(\frac{\varphi_\lambda u_r}{\varphi_\lambda^2} > 0\), this is possible only if \(\lambda_1(B(x_0, r)) \geq \lambda\).

To complete the proof we note that if \(r = \frac{1}{\lambda}\) then the metric is essentially Euclidean. We rescale the ball by \(x \to \lambda x\) (with coordinates centered at \(x_0\)) and then obtain an essentially Euclidean ball of radius \(r\). Then \(\lambda_1(B(x_0, \frac{1}{\lambda}) = \lambda \lambda_1 B_{g_0}(x_0, r)\). Therefore we only need to choose \(r\) so that \(\lambda_1 B_{g_0}(x_0, r) = 1\).

2.4. \textbf{Rectifiability of the nodal set.} We recall that the nodal set of an eigenfunction \(\varphi_\lambda\) is its zero set. When zero is a regular value of \(\varphi_\lambda\) the nodal set is a smooth hypersurface. This is a generic property of eigenfunctions [U]. It is pointed out in [Bae] that eigenfunctions can always be locally represented in the form

\[\varphi_\lambda(x) = v(x) \left( x_1^k + \sum_{j=0}^{k-1} x_1^j u_j(x') \right),\]

in suitable coordinates \((x_1, x')\) near \(p\), where \(\varphi_\lambda\) vanishes to order \(k\) at \(p\), where \(u_j(x')\) vanishes to order \(k - j\) at \(x' = 0\), and where \(v(x) \neq 0\) in a ball around \(p\). It follows that the nodal set is always countably \(n - 1\) rectifiable when \(\dim M = n\).
is the Riemannian surface measure, where \(dS\) denotes the Riemannian volume element on the nodal set, i.e. the insert \( \iota_n dV_g \) of the unit normal into the volume form of \((M, g)\).

The main result is:

**Theorem 3.1.** Let \((M, g)\) be a \(C^\infty\) Riemannian manifold. Then there exists a constant \(C\) independent of \(\lambda\) such that

\[
C \lambda^{1-\frac{n-1}{2}} \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}).
\]

We sketch the proof of Theorem 3.1 from SoZ, SoZa. The starting point is an identity from SoZ (inspired by an identity in Dong):

**Proposition 3.2.** For any \(f \in C^2(M)\),

\[
\int_M |\varphi_\lambda| (\Delta_g + \lambda^2) f dV_g = 2 \int_{Z_{\varphi_\lambda}} |\nabla_g \varphi_\lambda| f dS,
\]

When \(f \equiv 1\) we obtain

**Corollary 3.3.**

\[
\lambda^2 \int_M |\varphi_\lambda| dV_g = 2 \int_{Z_{\varphi_\lambda}} |\nabla_g \varphi_\lambda| f dS,
\]

**Exercise 3.** Prove this identity by decomposing \(M\) into a union of nodal domains.

The lower bound of Theorem 3.1 follows from the identity in Corollary 3.3 and the following lemma:

**Lemma 3.4.** If \(\lambda > 0\) then

\[
\|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}
\]

Here, \(A(\lambda) \lesssim B(\lambda)\) means that there exists a constant independent of \(\lambda\) so that \(A(\lambda) \leq CB(\lambda)\).

By Lemma 3.4 and Corollary 3.3, we have

\[
\lambda^2 \int_M |\varphi_\lambda| dV_g = 2 \int_{Z_{\varphi_\lambda}} |\nabla_g \varphi_\lambda| dS \lesssim 2 |Z_{\lambda}| \|\nabla_g \varphi_\lambda\|_{L^\infty(M)}
\]

Thus Theorem 3.1 follows from the somewhat curious cancellation of \(\|\varphi_\lambda\|_{L^1}\) from the two sides of the inequality.

**3.1. Proof of Lemma 3.4.**

Proof. The main idea is to construct a designer reproducing kernel for \(\varphi_\lambda\) of the form

\[
\hat{\rho}(\lambda - \sqrt{-\Delta_g}) f = \int_{-\infty}^{\infty} \rho(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f dt,
\]

with \(\rho \in C^\infty_0(\mathbb{R})\).
Exercise 4. Prove that (55) has the spectral expansion,

\[ \chi_\lambda f = \sum_{j=0}^{\infty} \hat{\rho}(\lambda - \lambda_j)E_j f, \]

where \( E_j f \) is the projection of \( f \) onto the \( \lambda_j \)-eigenspace of \( \sqrt{-\Delta_g} \). Conclude that (55) reproduces \( \varphi_\lambda \) if \( \hat{\rho}(0) = 1 \).

We denote the kernel of \( \chi_\lambda \) by \( K_\lambda(x, y) \), i.e.

\[ \chi_\lambda f(x) = \int_M K_\lambda(x, y)f(y)dV(y), \quad (f \in C(M)). \]

Assuming \( \hat{\rho}(0) = 1 \), then

\[ \int_M K_\lambda(x, y)\varphi_\lambda(y)dV(y) = \varphi_\lambda(x). \]

To obtain Lemma 3.4, we choose \( \rho \) so that the reproducing kernel \( K_\lambda(x, y) \) is uniformly bounded by \( \lambda^{-n/2} \) on the diagonal as \( \lambda \to +\infty \). It suffices to choose \( \rho \) so that \( \rho(t) = 0 \) for \( |t| \notin [\varepsilon/2, \varepsilon] \), with \( \varepsilon > 0 \) less than the injectivity radius of \((M, g)\).

Exercise 5. Prove that

\[ K_\lambda(x, y) = \lambda^{-n/2} a_\lambda(x, y)e^{i\lambda r(x, y)}, \]

where \( a_\lambda(x, y) \) is bounded with bounded derivatives in \((x, y)\) and where \( r(x, y) \) is the Riemannian distance between points. This WKB formula for \( K_\lambda(x, y) \) is known as a parametrix. (Hint: use the Hadamard parametrix) and stationary phase).

It follows from (57) that

\[ |\nabla_g K_\lambda(x, y)| \leq C\lambda^{1+n/2}, \]

and therefore,

\[ \sup_{x \in M} |\nabla_g \chi_\lambda f(x)| = \sup_x \left| \int f(y) \nabla_g K_\lambda(x, y)dV \right| \leq \|\nabla_g K_\lambda(x, y)\|_{L^\infty(M \times M)} \|f\|_{L^1} \leq C\lambda^{1+n/2} \|f\|_{L^1} \]

To complete the proof of Lemma 3.4, we set \( f = \varphi_\lambda \) and use that \( \chi_\lambda \varphi_\lambda = \varphi_\lambda \).

We view \( K_\lambda(x, y) \) as a designer reproducing kernel, because it is much smaller on the diagonal than kernels of the spectral projection operators \( E_{[\lambda, \lambda+1]} = \sum_{j: \lambda_j \in [\lambda, \lambda+1]} E_j \). The restriction on the support of \( \rho \) removes the big singularity on the diagonal at \( t = 0 \). As discussed in [SoZa], it is possible to use this kernel because we only need it to reproduce one eigenfunction and not a whole spectral interval of eigenfunctions.
3.2. Modifications. Hezari-Sogge modified the proof Proposition [HS] to prove

**Theorem 3.5.** For any $C^\infty$ compact Riemannian manifold, the $L^2$-normalized eigenfunctions satisfy

$$\mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \geq C \lambda \|\varphi_\lambda\|_{L^1}^2.$$  

They first apply the Schwarz inequality to get

$$\lambda^2 \int_M |\varphi_\lambda| dV_g \leq 2(\mathcal{H}^{n-1}(Z_{\varphi_\lambda}))^{1/2} \left( \int_{Z_{\varphi_\lambda}} |\nabla g\varphi|_{\varphi_\lambda}^2 dS \right)^{1/2}.$$  

They then use the test function

$$f = (1 + \lambda^2 \varphi_\lambda^2 + |\nabla g\varphi_{\lambda}|^2)^{1/2}$$  

in Proposition [HS] to show that

$$\int_{Z_{\varphi_\lambda}} |\nabla g\varphi_\lambda|_{\varphi_\lambda}^2 dS \leq \lambda^3.$$  

See also [Ar] for the generalization to the nodal bounds to Dirichlet and Neumann eigenfunctions of bounded domains.

Theorem 3.5 shows that Yau’s conjectured lower bound would follow for a sequence of eigenfunctions satisfying $||\varphi_\lambda||_{L^1} \geq C > 0$ for some positive constant $C$.

3.3. **Lower bounds on $L^1$ norms of eigenfunctions.** The following universal lower bound is optimal as $(M, g)$ ranges over all compact Riemannian manifolds.

**Proposition 15.** For any $(M, g)$ and any $L^2$-normalized eigenfunction, $||\varphi_\lambda||_{L^1} \geq C_g \lambda^{-\frac{n-1}{2}}$.

**Remark:** There are few results on $L^1$ norms of eigenfunctions. The reason is probably that $|\varphi_\lambda|^2 dV$ is the natural probability measure associated to eigenfunctions. It is straightforward to show that the expected $L^1$ norm of random $L^2$-normalized spherical harmonics of degree $N$ and their generalizations to any $(M, g)$ is a positive constant $C_N$ with a uniform positive lower bound. One expects eigenfunctions in the ergodic case to have the same behavior.

**Problem 1.** A difficult but interesting problem would be to show that $||\varphi_\lambda||_{L^1} \geq C_0 > 0$ on a compact hyperbolic manifold. A partial result in this direction would be useful.

3.4. **Dong’s upper bound.** Let $(M, g)$ be a compact $C^\infty$ Riemannian manifold of dimension $n$, let $\varphi_\lambda$ be an $L^2$-normalized eigenfunction of the Laplacian,

$$\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda.$$  

Let

$$q = |\nabla \varphi|^2 + \lambda^2 \varphi^2.$$  

In Theorem 2.2 of [D], R. T. Dong proves the bound

$$\mathcal{H}^{n-1}(\mathcal{N} \cap \Omega) \leq \frac{1}{2} \int_{\Omega} |\nabla \log q| + \sqrt{\text{vol}(\Omega)} \lambda + \text{vol}(\partial \Omega).$$
He also proves (Theorem 3.3) that on a surface,

\[ \Delta \log q \geq -\lambda + 2 \min(K, 0) + 4\pi \sum_i (k_i - 1)\delta_{p_i}, \]

where \( \{p_i\} \) are the singular points and \( k_i \) is the order of \( p_i \). In Dong's notation, \( \lambda > 0 \). Using a weak Harnack inequality, Dong shows (loc. cit. Theorem 4.2) how (65) and (64) combine to produce the upper bound \( H^1(N \cap \Omega) \leq \lambda^{3/2} \) in dimension 2.

**Problem 2.** To what extent can one generalize these estimates to higher dimensions?

### 3.5. Other level sets.

Although nodal sets are special, it is of interest to bound the Hausdorff surface measure of any level set \( N_c^{\varphi_\lambda} := \{\varphi_\lambda = c\} \). Let \( \text{sgn}(x) = \frac{x}{|x|} \).

**Proposition 3.6.** For any \( C^\infty \) Riemannian manifold, and any \( f \in C(M) \) we have,

\[ \int_M f(\Delta + \lambda^2) |\varphi_\lambda - c| dV + \lambda^2 c \int f \text{sgn}(\varphi_\lambda - c) dV = 2 \int_{N_c^{\varphi_\lambda}} f|\nabla \varphi_\lambda| dS. \]

This identity has similar implications for \( H^{n-1}(N_c^{\varphi_\lambda}) \) and for the equidistribution of level sets.

**Corollary 3.7.** For \( c \in \mathbb{R} \)

\[ \lambda^2 \int_{\varphi_\lambda \geq c} \varphi_\lambda dV = \int_{N_c^{\varphi_\lambda}} |\nabla \varphi_\lambda| dS. \]

One can obtain lower bounds on \( H^{n-1}(N_c^{\varphi_\lambda}) \) as in the case of nodal sets. However the integrals of \( |\varphi_\lambda| \) no longer cancel out. The numerator is smaller since one only integrates over \( \{\varphi_\lambda \geq c\} \). Indeed, \( H^{n-1}(N_c^{\varphi_\lambda}) \) must tend to zero as \( c \) tends to the maximum possible threshold \( \lambda \frac{n-1}{2} \) for \( \sup_M |\varphi_\lambda| \).

The Corollary follows by integrating \( \Delta \) by parts, and by using the identity,

\[ \int_M |\varphi_\lambda - c| + c \text{sgn}(\varphi_\lambda - c) \ dV = \int_{\varphi_\lambda > c} \varphi_\lambda dV - \int_{\varphi_\lambda < c} \varphi_\lambda dV \]

\[ = 2 \int_{\varphi_\lambda > c} \varphi_\lambda dV, \]

since \( 0 = \int_M \varphi_\lambda dV = \int_{\varphi_\lambda > c} \varphi_\lambda dV + \int_{\varphi_\lambda < c} \varphi_\lambda dV. \)

**Problem 3.** A difficult problem would be to study \( H^{n-1}(N_c^{\varphi_\lambda}) \) as a function of \((c, \lambda)\) and try to find thresholds where the behavior changes. For random spherical harmonics, \( \sup_M |\varphi_\lambda| \simeq \sqrt{\log \lambda} \) and one would expect the level set volumes to be very small above this height except in special cases.

### 3.6. Examples.

The lower bound of Theorem 3.1 is far from the lower bound conjectured by Yau, which by Theorem 3.1 is correct at least in the real analytic case. In this section we go over the model examples to understand why the methods are not always getting sharp results.
3.6.1. Flat tori. We have, $|\nabla \sin(k, x)|^2 = \cos^2(k, x)|k|^2$. Since $\cos(k, x) = 1$ when $\sin(k, x) = 0$ the integral is simply $|k|$ times the surface volume of the nodal set, which is known to be of size $|k|$. Also, we have $\int_{\mathcal{T}} |\sin(k, x)| dx \geq C$. Thus, our method gives the sharp lower bound $\mathcal{H}^{n-1}(Z_{\varphi}) \geq C\lambda^1$ in this example.

So the upper bound is achieved in this example. Also, we have $\int_{\mathcal{T}} |\sin(k, x)| dx \geq C$. Thus, our method gives the sharp lower bound $\mathcal{H}^{n-1}(Z_{\varphi}) \geq C\lambda^1$ in this example. Since $\cos(k, x) = 1$ when $\sin(k, x) = 0$ the integral is simply $|k|$ times the surface volume of the nodal set, which is known to be of size $|k|$.

3.6.2. Spherical harmonics on $S^2$. For background on spherical harmonics we refer to §11.

The $L^1$ of $Y_0^n$ norm can be derived from the asymptotics of Legendre polynomials

$$P_N(\cos \theta) = \sqrt{2/(\pi N)} \sin \theta - \frac{1}{2} \cos \left(\frac{N+1}{2} \theta - \frac{\pi}{4}\right) + O(N^{-3/2})$$

where the remainder is uniform on any interval $\epsilon < \theta < \pi - \epsilon$. We have

$$||Y_0^n||_{L^1} = 4\pi \sqrt{\frac{(2N+1)}{2\pi}} \int_0^{\pi/2} |P_N(\cos \theta)| dv(\theta) \sim C_0 > 0,$$

e.i. the $L^1$ norm is asymptotically a positive constant. Hence $\int_{Z_{Y_0^n}} |\nabla Y_0^n| ds \simeq C_0 N^2$. In this example $|\nabla Y_0^n|_{L^\infty} = N^{\frac{3}{2}}$ saturates the sup norm bound. The length of the nodal line of $Y_0^n$ is of order $\lambda$, as one sees from the rotational invariance and by the fact that $P_N$ has $N$ zeros. The defect in the argument is that the bound $|\nabla Y_0^n|_{L^\infty} = N^{\frac{3}{2}}$ is only obtained on the nodal components near the poles, where each component has length $\simeq \frac{1}{N}$.

**Exercise 6.** Calculate the $L^1$ norms of (L2-normalized) zonal spherical harmonics and Gaussian beams.

The left image is a zonal spherical harmonic of degree $N$ on $S^2$: it has high peaks of height $\sqrt{N}$ at the north and south poles. The right image is a Gaussian beam: its height along the equator is $N^{1/4}$ and then it has Gaussian decay transverse to the equator.

**Gaussian beams**

Gaussian beams are Gaussian shaped lumps which are concentrated on $\lambda^{-\frac{1}{2}}$ tubes $\mathcal{T}_{\lambda^{-\frac{1}{2}}} (\gamma)$ around closed geodesics and have height $\lambda^{\frac{n-1}{2}}$. We note that their $L^1$ norms decrease like $\lambda^{-(n+1)}$, i.e. they saturate the $L^p$ bounds of [Sog] for small $p$. In such cases we have $\int_{Z_{\varphi}} |\nabla \varphi| dS \simeq \lambda^2 ||\varphi||_{L^1} \simeq \lambda^{2-\frac{n-1}{2}}$. It is likely that Gaussian beams are minimizers of the $L^1$ norm among L2-normalized eigenfunctions of Riemannian manifolds. Also, the gradient bound $||\nabla \varphi||_{L^\infty} = O(\lambda^{\frac{n+1}{2}})$ is far off for Gaussian beams, the correct upper bound being $\lambda^{1+\frac{n-1}{2}}$. If we use these estimates on $||\varphi||_{L^1}$ and $||\nabla \varphi||_{L^\infty}$, our method gives $\mathcal{H}^{n-1}(Z_{\varphi}) \geq C\lambda^{-\frac{n-1}{2}}$, while $\lambda$ is the correct lower bound for Gaussian beams in
the case of surfaces of revolution (or any real analytic case). The defect is again that the gradient estimate is achieved only very close to the closed geodesic of the Gaussian beam. Outside of the tube $T_{\lambda^{-\frac{1}{2}}}(\gamma)$ of radius $\lambda^{-\frac{1}{2}}$ around the geodesic, the Gaussian beam and all of its derivatives decay like $e^{-\lambda d^2}$ where $d$ is the distance to the geodesic. Hence $\int_Z |\nabla \varphi_\lambda| dS \simeq \int_{Z_{\varphi_\lambda} \cap T_{\lambda^{-\frac{1}{2}}}(\gamma)} |\nabla \varphi_\lambda| dS$. Applying the gradient bound for Gaussian beams to the latter integral gives $H^{n-1}(Z_{\varphi_\lambda} \cap T_{\lambda^{-\frac{1}{2}}}(\gamma)) \geq C \lambda^{1-\frac{n}{2}}$, which is sharp since the intersection $Z_{\varphi_\lambda} \cap T_{\lambda^{-\frac{1}{2}}}(\gamma)$ cuts across $\gamma$ in $\simeq \lambda$ equally spaced points (as one sees from the Gaussian beam approximation).

4. Analytic continuation of eigenfunctions to the complex domain

We next discuss three results that use analytic continuation of eigenfunctions to the complex domain. First is the Donnelly-Fefferman volume bound Theorem \(\text{[Ze6]}\). We sketch a somewhat simplified proof which will appear in more detail in \(\text{[Ze0]}\). Second we discuss the equidistribution theory of nodal sets in the complex domain in the ergodic case \(\text{[Ze7]}\) and in the completely integrable case \(\text{[Ze8]}\). Third, we discuss nodal intersection bounds. This includes bounds on the number of nodal lines intersecting the boundary in \(\text{[TZ]}\) for the Dirichlet or Neuman problem in a plane domain, the number (and equi-distribution) of nodal intersections with geodesics in the complex domain \(\text{[Ze0]}\) and results on nodal intersections and nodal domains for the modular surface.

4.1. Grauert tubes. . As examples, we have:

- $M = \mathbb{R}^m / \mathbb{Z}^m$ is $M_C = \mathbb{C}^m / \mathbb{Z}^m$.
- The unit sphere $S^n$ defined by $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$ is complexified as the complex quadric $S_C^n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1 \}$.
- The hyperboloid model of hyperbolic space is the hypersurface in $\mathbb{R}^{n+1}$ defined by $H^n = \{ x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, \ x_n > 0 \}$.

Then, $H_C^n = \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_n^2 - z_{n+1}^2 = -1 \}$.

- Any real algebraic subvariety of $\mathbb{R}^m$ has a similar complexification.
- Any Lie group $G$ (or symmetric space) admits a complexification $G_C$.

Let us consider examples of holomorphic continuations of eigenfunctions:

- On the flat torus $\mathbb{R}^m / \mathbb{Z}^m$, the real eigenfunctions are $\cos\langle k, x \rangle, \sin\langle k, x \rangle$ with $k \in 2\pi \mathbb{Z}^m$. The complexified torus is $\mathbb{C}^m / \mathbb{Z}^m$ and the complexified eigenfunctions are $\cos\langle k, \zeta \rangle, \sin\langle k, \zeta \rangle$ with $\zeta = x + i \xi$.

- On the unit sphere $S^m$, eigenfunctions are restrictions of homogeneous harmonic functions on $\mathbb{R}^{m+1}$. The latter extend holomorphically to holomorphic harmonic polynomials on $\mathbb{C}^{m+1}$ and restrict to holomorphic function on $S^m_C$.

- On $H^m$, one may use the hyperbolic plane waves $e^{i(k+1)(z,b)}$, where $(z, b)$ is the (signed) hyperbolic distance of the horocycle passing through $z$ and $b$ to 0. They may be holomorphically extended to the maximal tube of radius $\pi/4$. 


On compact hyperbolic quotients $H^m/\Gamma$, eigenfunctions can be then represented by Helgason’s generalized Poisson integral formula $\mathcal{H}$,

$$\varphi_\lambda(z) = \int_B e^{(i\lambda+1)\langle z,b \rangle} dT_\lambda(b).$$

Here, $z \in D$ (the unit disc), $B = \partial D$, and $dT_\lambda \in \mathcal{D}'(B)$ is the boundary value of $\varphi_\lambda$, taken in a weak sense along circles centered at the origin $0$. To analytically continue $\varphi_\lambda$ it suffices to analytically continue $\langle z,b \rangle$. Writing the latter as $\langle \zeta,b \rangle$, we have:

$$\varphi^c_\lambda(\zeta) = \int_B e^{(i\lambda+1)\langle \zeta,b \rangle} dT_\lambda(b). \tag{68}$$

The modulus squares

$$|\varphi^c_j(\zeta)|^2 : M_\epsilon \to \mathbb{R}_+ \tag{69}$$

are sometimes known as Husimi functions. They are holomorphic extensions of $L^2$-normalized functions but are not themselves $L^2$ normalized on $M_\epsilon$. However, as will be discussed below, their $L^2$ norms may on the Grauert tubes (and their boundaries) can be determined. One can then ask how the mass of the normalized Husimi function is distributed in phase space, or how the $L^p$ norms behave.

### 4.2. Weak * limit problem for Husimi measures in the complex domain.

Find all of the weak* limits of the sequence,

$$\left\{ \frac{|\varphi^c_j(z)|^2}{\|\varphi^c_j\|_{L^2(\partial M_\epsilon)}} d\mu_\epsilon \right\}_{j=1}^\infty.$$

### 4.3. Poincaré-Lelong formula.

One of the two key reasons for the gain in simplicity is that there exists a simple analytical formula for the delta-function on the nodal set. The Poincaré-Lelong formula gives an exact formula for the delta-function on the zero set of $\varphi_j$

$$i\partial\bar{\partial} \log |\varphi^c_j(z)|^2 = [N\varphi^c_j]. \tag{70}$$

Thus, if $\psi$ is an $(n-1,n-1)$ form,
\[ \int_{N \phi^C_J} \psi = \int_{M_\epsilon} \psi \wedge i\partial\bar{\partial} \log |\phi^C_J(z)|^2. \]

4.4. Pluri-subharmonic functions and compactness. In the real domain, we have emphasized the problem of finding weak* limits of the probability measures (13) and of their microlocal lifts or Wigner measures in phase space. The same problem exists in the complex domain for the sequence of Husimi functions (69). However, there also exists a new problem involving the sequence of normalized logarithms

\[ \{ u_j := \frac{1}{\lambda_j} \log |\phi^C_j(z)|^2 \} \to G \]  

A key fact is that this sequence is pre-compact in \( L^p(M_\epsilon) \) for all \( p < \infty \) and even that

\[ \{ \frac{1}{\lambda_j} \nabla \log |\phi^C_j(z)|^2 \} \to G \]  

is pre-compact in \( L^1(M_\epsilon) \).

**HARTOGS Lemma.** (Hartog's Lemma; (see HoI, Theorem 4.1.9)): Let \( \{ v_j \} \) be a sequence of subharmonic functions in an open set \( X \subset \mathbb{R}^m \) which have a uniform upper bound on any compact set. Then either \( v_j \to -\infty \) uniformly on every compact set, or else there exists a subsequence \( v_{j_k} \) which is convergent to some \( u \in L^1_{\text{loc}}(X) \). Further, \( \limsup_{n} u_n(x) \leq u(x) \) with equality almost everywhere. For every compact subset \( K \subset X \) and every continuous function \( f \),

\[ \limsup_{n} \sup_{K} (u_n - f) \leq \sup_{K} (u - f). \]

In particular, if \( f \geq u \) and \( \epsilon > 0 \), then \( u_n \leq f + \epsilon \) on \( K \) for \( n \) large enough.

4.5. A general weak* limit problem. The study of exponential growth rates gives rise to a new kind new weak* limit problem for complexified eigenfunctions.

**Problem 4.2.** Find the weak* limits \( G \) on \( M_\epsilon \) of sequences

\[ \frac{1}{\lambda_{j_k}} \log |\phi^C_{j_k}(z)|^2 \to G? \]

( The limits are actually in \( L^1 \) and not just weak. )

Here is a general Heuristic principle to pin down the possible \( G \): If \( \frac{1}{\lambda_{j_k}} \log |\phi^C_{j_k}(z)|^2 \to G(z) \) then

\[ |\phi^C_{j_k}(z)|^2 \approx e^{\lambda_{j_k} G(z)} (1 + \text{SOMETHING SMALLER}) \ (\lambda_{j_k} \to \infty). \]

But \( \Delta_C |\phi^C_{j_k}(z)|^2 = \lambda_{j_k}^2 |\phi^C_{j_k}(z)|^2 \), so we should have

**Conjecture 4.3.** Any limit \( G \) as above solves the Hamilton-Jacobi equation,

\[ (\nabla_C G)^2 = 1. \]

(Note: The weak* limits of \( \frac{|\phi^C_j(z)|^2}{\|\phi^C_j\|_{L^2(\partial M_\epsilon)}} \ d\mu_\epsilon \) must be supported in \( \{ G = G_{\text{max}} \} \) (i.e. in the set of maximum values).
5. Poisson operator and Szegő operators on Grauert tubes

5.1. Poisson operator and analytic Continuation of eigenfunctions. The half-wave group of \((M, g)\) is the unitary group \(U(t) = e^{it\sqrt{\Delta}}\) generated by the square root of the positive Laplacian. Its Schwartz kernel is a distribution on \(\mathbb{R} \times M \times M\) with the eigenfunction expansion

\[
U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x)\varphi_j(y).
\]

By the Poisson operator we mean the analytic continuation of \(U(t)\) to positive imaginary time,

\[
e^{-t\sqrt{\Delta}} = U(it).
\]

The eigenfunction expansion then converges absolutely to a real analytic function on \(\mathbb{R}_+ \times M \times M\).

Let \(A(\tau)\) denote the operator of analytic continuation of a function on \(M\) to the Grauert tube \(M_\tau\). Since

\[
U_C(i\tau)\varphi_\lambda = e^{-\tau\lambda} \varphi_\lambda,
\]

it is simple to see that

\[
A(\tau) = U_C(i\tau)e^{\tau\sqrt{\Delta}}
\]

where \(U_C(i\tau, \zeta, y)\) is the analytic continuation of the Poisson kernel in \(x\) to \(M_\tau\). In terms of the eigenfunction expansion, one has

\[
U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^C(\zeta)\varphi_j(y), \quad (\zeta, y) \in M_\epsilon \times M.
\]

This is a very useful observation because \(U_C(i\tau)e^{\tau\sqrt{\Delta}}\) is a Fourier integral operator with complex phase and can be related to the geodesic flow. The analytic continuability of the Poisson operator to \(M_\tau\) implies that every eigenfunction analytically continues to the same Grauert tube.

5.2. Analytic continuation of the Poisson wave group. The analytic continuation of the Poisson-wave kernel to \(M_\tau\) in the \(x\) variable is discussed in detail in [Ze8] and ultimately derives from the analysis by Hadamard of his parametrix construction. We only briefly discuss it here and refer to [Ze8] for further details. In the case of Euclidean \(\mathbb{R}^n\) and its wave kernel \(U(t, x, y) = \int_{\mathbb{R}^n} e^{it||\xi||} e^{i\langle\xi, x-y\rangle} d\xi\) which analytically continues to \(t + i\tau, \zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n\) as the integral

\[
U_C(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+ir)||\xi||} e^{i\langle\xi, x+ip-y\rangle} d\xi.
\]

The integral clearly converges absolutely for \(|p| < \tau\).

Exact formulae of this kind exist for \(S^m\) and \(H^m\). For a general real analytic Riemannian manifold, there exists an oscillatory integral expression for the wave kernel of the form,

\[
U(t, x, y) = \int_{T^*_y M} e^{it||\xi||} e^{i\langle\xi, \exp^{-1}(x)\rangle} A(t, x, \xi) d\xi
\]
where \( A(t, x, y, \xi) \) is a polyhomogeneous amplitude of order 0. The holomorphic extension of (78) to the Grauert tube \(|\zeta| < \tau\) in \( x \) at time \( t = i\tau \) then has the form

\[
U_C(i\tau, \zeta, y) = \int_{T^*_y} e^{-\tau|\xi|} e^{i\xi, \text{exp}^{-1}(\zeta)} A(t, \zeta, y, \xi) d\xi \quad (\zeta = x + iy).
\]  

### 5.3. Complexified spectral projections.

The next step is to holomorphically extend the spectral projectors \( d\Pi_{[0, \lambda]}(x, y) = \sum_j \delta(\lambda - \lambda_j)\varphi_j(x)\varphi_j(y) \) of \( \sqrt{\Delta} \). The complexified diagonal spectral projections measure is defined by

\[
d_{\lambda} \Pi^C_{[0, \lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j)|\varphi^C_j(\zeta)|^2.
\]

Henceforth, we generally omit the superscript and write the kernel as \( \Pi^C_{[0, \lambda]}(\zeta, \bar{\zeta}) \). This kernel is not a tempered distribution due to the exponential growth of \( |\varphi^C_j(\zeta)|^2 \). Since many asymptotic techniques assume spectral functions are of polynomial growth, we simultaneously consider the damped spectral projections measure

\[
d_{\lambda} P^c_{[0, \lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j)e^{-2\tau \lambda_j}|\varphi^C_j(\zeta)|^2,
\]

which is a temperate distribution as long as \( \sqrt{\rho}(\zeta) \leq \tau \). When we set \( \tau = \sqrt{\rho}(\zeta) \) we omit the \( \tau \) and put

\[
d_{\lambda} P^c_{[0, \lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j)e^{-2\sqrt{\rho}(\zeta) \lambda_j}|\varphi^C_j(\zeta)|^2.
\]

The integral of the spectral measure over an interval \( I \) gives

\[
\Pi_I(x, y) = \sum_{j: \lambda_j \in I} \varphi_j(x)\varphi_j(y).
\]

Its complexification gives the spectral projections kernel along the anti-diagonal,

\[
\Pi^C_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} |\varphi^C_j(\zeta)|^2,
\]

and the integral of (81) gives its temperate version

\[
P^c_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\sqrt{\rho}(\zeta) \lambda_j}|\varphi^C_j(\zeta)|^2,
\]

or in the crucial case of \( \tau = \sqrt{\rho}(\zeta) \),

\[
P^c_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\sqrt{\rho}(\zeta) \lambda_j}|\varphi^C_j(\zeta)|^2.
\]
5.4. Poisson operator as a complex Fourier integral operator. The damped spectral projection measure $d_\lambda P_{[\alpha,\lambda]}^{\partial M}(\zeta,\bar{\zeta})$ (51) is dual under the real Fourier transform in the $t$ variable to the restriction

$$U(t + 2i\tau, \zeta, \bar{\zeta}^\prime) = \sum_j e^{(-2\tau + it)\lambda_j} |\varphi^C_j(\zeta)|^2$$

to the anti-diagonal of the mixed Poisson-wave group. The adjoint of the Poisson kernel $U(it, x, y)$ also admits an anti-holomorphic extension in the $y$ variable. The sum (86) are the diagonal values of the complexified wave kernel

$$U(t + 2i\tau, \zeta, \bar{\zeta}^\prime) = \int_M U(t + i\tau, \zeta, y) E(i\tau, y, \bar{\zeta}^\prime) dV_g(x)$$

where $E(t)$ is the sum (86) of eigenfunctions on $M$. The adjoint of the Poisson kernel $U(it, x, y)$ also admits an anti-holomorphic extension in the $y$ variable. The sum (86) are the diagonal values of the complexified wave kernel

$$U(t + 2i\tau, \zeta, \bar{\zeta}^\prime) = \sum_j e^{(-2\tau + it)\lambda_j} \varphi^C_j(\zeta) \varphi^C_j(\zeta^\prime).$$

We obtain (87) by orthogonality of the real eigenfunctions on $M$.

Since $U(t + 2i\tau, \zeta, y)$ takes its values in the CR holomorphic functions on $\partial M$, we consider the Sobolev spaces $\mathcal{O}^{s+\frac{m-1}{4}}(\partial M)$ of CR holomorphic functions on the boundaries of the strictly pseudo-convex domains $M_\epsilon$, i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial M) = W^{s+\frac{m-1}{4}}(\partial M) \cap \mathcal{O}(\partial M),$$

where $W_s$ is the $s$th Sobolev space and where $\mathcal{O}(\partial M)$ is the space of boundary values of holomorphic functions. The inner product on $\mathcal{O}^0(\partial M)$ is with respect to the Liouville measure

$$d\mu = (i\partial \bar{\partial} \sqrt{\rho})^{m-1} \wedge \sqrt{\rho}.$$  

We then regard $U(t + i\tau, \zeta, y)$ as the kernel of an operator from $L^2(M) \to \mathcal{O}^0(\partial M)$. It equals its composition $\Pi_\tau \circ U(t + i\tau)$ with the Szeg"o projector

$$\Pi_\tau : L^2(\partial M) \to \mathcal{O}^0(\partial M)$$

for the tube $M_\epsilon$, i.e. the orthogonal projection onto boundary values of holomorphic functions in the tube.

This is a useful expression for the complexified wave kernel, because $\Pi_\tau$ is a complex Fourier integral operator with a small wave front relation. More precisely, the real points of its canonical relation form the graph $\Delta^\tau$ of the identity map on the symplectic one $\Sigma_\tau \subset T^*\partial M$ spanned by the real one-form $d^\alpha \rho$, i.e.

$$\Sigma_\tau = \{ (\zeta; rd^\alpha \rho(\zeta)) : \zeta \in \partial M_\tau, \ r > 0 \} \subset T^*\partial M_\tau.$$  

We note that for each $\tau$, there exists a symplectic equivalence $\Sigma_\tau \simeq T^*\Sigma$ by the map $(\zeta, rd^\alpha \rho(\zeta)) \to (E^{-1}_\tau(\zeta), r\alpha)$, where $\alpha = \zeta \cdot dx$ is the action form (cf. [GS2]).

The following result was first stated by Boutet de Monvel [Bou] and has been proved in detail in [Zes, L, Ste].

**Theorem 5.1.** $\Pi_\epsilon \circ U(i\epsilon) : L^2(M) \to \mathcal{O}(\partial M_\epsilon)$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ associated to the canonical relation

$$\Gamma = \{ (y, \eta, \iota_\epsilon(y, \eta)) \} \subset T^*\Sigma_\epsilon.$$  

Moreover, for any $s$,

$$\Pi_\epsilon \circ U(i\epsilon) : W^s(M) \to \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\epsilon)$$
is a continuous isomorphism.

In \cite{Ze8} we give the following sharpening of the sup norm estimates of \cite{Bou}:

**Proposition 5.2.** Suppose \((M, g)\) is real analytic. Then

\[
\sup_{\zeta \in M} |\varphi^c_\lambda(\zeta)| \leq C \lambda^{\frac{m+1}{2}} e^{\tau \lambda}, \quad \sup_{\zeta \in M} \left| \frac{\partial \varphi^c_\lambda(\zeta)}{\partial \zeta_j} \right| \leq C \lambda^{\frac{m+1}{2}} e^{\tau \lambda}
\]

The proof follows easily from the fact that the complexified Poisson kernel is a complex Fourier integral operator of finite order. The estimates can be improved further.

5.5. **Toeplitz dynamical construction of the wave group.** There exists an alternative to the parametrix constructions of Hadamard-Riesz, Lax, Hörmander and others which are reviewed in \S 12. It is useful for constructing the wave group \(U(t)\) for large \(t\), when it is awkward to use the group property \(U(t/N)^N = U(t)\). As in Theorem 5.1 we denote by \(U(i\epsilon)\) the operator with kernel \(U(i\epsilon, \zeta, y)\) with \(\zeta \in \partial M_\epsilon, y \in M\). We also denote by \(U^*(i\epsilon) : \mathcal{O}(\partial M_\epsilon) \rightarrow L^2(M)\) the adjoint operator. Further, let

\[T_{g^t} : L^2(\partial M_\epsilon, d\mu_\epsilon) \rightarrow L^2(\partial M_\epsilon, d\mu_\epsilon)\]

be the unitary translation operator

\[T_{g^t}f(\zeta) = f(g^t(\zeta))\]

where \(d\mu_\epsilon\) is the contact volume form on \(\partial M_\epsilon\) and \(g^t\) is the Hamiltonian flow of \(\sqrt{\rho}\) on \(M_\epsilon\).

**Proposition 5.3.** There exists a symbol \(\sigma_{\epsilon, t}\) such that

\[U(t) = U^*(i\epsilon)\sigma_{\epsilon, t}T_{g^t}U(i\epsilon).\]

The proof of this Proposition is to verify that the right side is a Fourier integral operator with canonical relation the graph of the geodesic flow. One then constructs \(\sigma_{\epsilon, t}\) so that the symbols match. The proof is given in \cite{Ze6}. Related constructions are given in \cite{G1, BoGu}.

6. **Equidistribution of complex nodal sets of real ergodic eigenfunctions on analytic \((M, g)\) with ergodic geodesic flow**

We now consider global results when hypotheses are made on the dynamics of the geodesic flow. Use of the global wave operator brings into play the relation between the geodesic flow and the complexified eigenfunctions, and this allows one to prove goblal results on nodal hypersurfaces that reflect the dynamics of the geodesic flow. In some cases, one can determine not just the volume, but the limit distribution of complex nodal hypersurfaces. Since we have discussed this result elsewhere \cite{Ze6} we only briefly review it here.

The complex nodal hypersurface of an eigenfunction is defined by

\[(90) \quad Z_{\varphi^c_\lambda} = \{ \zeta \in B^*_\epsilon M : \varphi^c_\lambda(\zeta) = 0 \}.
\]

There exists a natural current of integration over the nodal hypersurface in any ball bundle \(B^*_\epsilon M\) with \(\epsilon < \epsilon_0\), given by

\[(91) \quad \langle [Z_{\varphi^c_\lambda}], \varphi \rangle = \frac{i}{2\pi} \int_{B^*_\epsilon M} \partial \overline{\partial} \log |\varphi^c_\lambda|^2 \wedge \varphi = \int_{Z_{\varphi^c_\lambda}} \varphi, \quad \varphi \in \mathcal{D}^{(m-1,m-1)}(B^*_\epsilon M).
\]
In the second equality we used the Poincaré-Lelong formula. The notation \( D^{(m-1,m-1)}(B^*_\epsilon M) \) stands for smooth test \((m-1,m-1)\)-forms with support in \( B^*_\epsilon M \).

The nodal hypersurface \( Z_{\varphi^C_\lambda} \) also carries a natural volume form \(|Z_{\varphi^C_\lambda}|\) as a complex hypersurface in a Kähler manifold. By Wirtinger’s formula, it equals the restriction of \( \frac{\omega_y^{m-1}}{(m-1)!} \) to \( Z_{\varphi^C_\lambda} \). Hence, one can regard \( Z_{\varphi^C_\lambda} \) as defining the measure

\[
\langle |Z_{\varphi^C_\lambda}|, \varphi \rangle = \int_{Z_{\varphi^C_\lambda}} \varphi \frac{\omega_y^{m-1}}{(m-1)!}, \quad \varphi \in C(B^*_\epsilon M).
\]

We prefer to state results in terms of the current \( [Z_{\varphi^C_\lambda}] \) since it carries more information.

**Theorem 6.1.** Let \((M,g)\) be real analytic, and let \{\(\varphi_{jk}\)\} denote a quantum ergodic sequence of eigenfunctions of its Laplacian \(\Delta\). Let \((B^*_\epsilon M, J)\) be the maximal Grauert tube around \(M\) with complex structure \(J_g\) adapted to \(g\). Let \(\epsilon < \epsilon_0\). Then:

\[
\frac{1}{\lambda_{jk}} [Z_{\varphi^C_{jk}}] \to \frac{i}{\pi} \partial \bar{\partial} \sqrt{\rho} \text{ weakly in } D^{(1,1)}(B^*_\epsilon M),
\]

in the sense that, for any continuous test form \(\psi \in D^{(m-1,m-1)}(B^*_\epsilon M)\), we have

\[
\frac{1}{\lambda_{jk}} \int_{Z_{\varphi^C_{jk}}} \psi \to \frac{i}{\pi} \int_{B^*_\epsilon M} \psi \wedge \partial \bar{\partial} \sqrt{\rho}.
\]

Equivalently, for any \(\varphi \in C(B^*_\epsilon M)\),

\[
\frac{1}{\lambda_{jk}} \int_{Z_{\varphi^C_{jk}}} \varphi \frac{\omega_y^{m-1}}{(m-1)!} \to \frac{i}{\pi} \int_{B^*_\epsilon M} \varphi \partial \bar{\partial} \sqrt{\rho} \wedge \frac{\omega_y^{m-1}}{(m-1)!}.
\]

A key input is the following quantum ergodicity theorem in the complex domain in [Zel5].

**Theorem 6.2.** If the geodesic flow is ergodic, then for all but a sparse subsequence of \(\lambda_j\),

\[
\frac{1}{\lambda_{jk}} \log |\varphi_{jk}^C(z)|^2 \to \sqrt{\rho} \text{ in } L^1(M_\epsilon).
\]

This is the maximum possible growth rate: it says that ergodic eigenfunctions have the maximum exponential growth rate possible for any eigenfunctions.

**Corollary 6.3.** Let \((M,g)\) be a real analytic with ergodic geodesic flow. Let \{\(\varphi_{jk}\)\} denote a full density ergodic sequence. Then for all \(\epsilon < \epsilon_0\),

\[
\frac{1}{\lambda_{jk}} [Z_{\varphi^C_{jk}}] \to \frac{i}{\pi} \partial \bar{\partial} \sqrt{\rho}, \text{ weakly in } D^{(1,1)}(B^*_\epsilon M).
\]

The proof consists of three ingredients:

1. By the Poincaré-Lelong formula, \([Z_{\varphi^C_\lambda}] = i \partial \bar{\partial} \log |\varphi^C_\lambda|\). This reduces the theorem to determining the limit of \(\frac{1}{\lambda} \log |\varphi^C_\lambda|\).

2. \(\frac{1}{\lambda} \log |\varphi^C_\lambda|\) is a sequence of PSH functions which are uniformly bounded above by \(\sqrt{\rho}\). By a standard compactness theorem, the sequence is pre-compact in \(L^1\): every sequence from the family has an \(L^1\) convergent subsequence.
(3) $|\varphi_{x}^{\epsilon}|^2$, when properly $L^2$ normalized on each $\partial M_\tau$ is a quantum ergodic sequence on $\partial M_\tau$. This property implies that the $L^2$ norm of $|\varphi_{x}^{\epsilon}|^2$ on $\partial \Omega$ is asymptotically $\sqrt{p}$.

(4) Ergodicity and the calculation of the $L^2$ norm imply that the only possible $L^1$ limit of $\frac{1}{\lambda} \log |\varphi_{x}^{\epsilon}|$. This concludes the proof.

We note that the first two steps are valid on any real analytic $(M, \varrho)$. The difference is that the $L^2$ norms of $\varphi_{x}^{\epsilon}$ may depend on the subsequence and can often not equal $\sqrt{p}$. That is, $\frac{1}{\lambda} |\varphi_{x}^{\epsilon}|$ behaves like the maximal PSH function in the ergodic case, but not in general.

For instance, on a flat torus, the complex zero sets of ladders of eigenfunctions concentrate that the $L^2$ manifold confirms the upper bound on complex nodal hypersurface volumes.

We now give more details. A key object in the proof is the sequence of functions $U_\lambda(x, \xi) \in C^\infty(B_\epsilon^* M)$ defined by

$$\begin{align*}
U_\lambda(x, \xi) := & \frac{\varphi_{x}^{\epsilon(x,\xi)}}{\rho_{\lambda}(x,\xi)}, \quad (x, \xi) \in B_\epsilon^* M, \quad \text{where} \\
\rho_{\lambda}(x, \xi) := & ||\varphi_{x}^{\epsilon}\rangle_{\partial B_{|\xi|\varrho}}||_{L^2(\partial B_{|\xi|\varrho}^* M)}
\end{align*}$$

Thus, $\rho_{\lambda}(x, \xi)$ is the $L^2$-norm of the restriction of $\varphi_{x}^{\epsilon}$ to the sphere bundle $\{\partial B_{|\xi|\varrho}^* M\}$ where $\epsilon = |\xi|\varrho$. $U_\lambda$ is of course not holomorphic, but its restriction to each sphere bundle is CR holomorphic there, i.e.

$$u_{\lambda} = U_\lambda|_{\partial B_{|\xi|\varrho}^* M} \in \mathcal{O}^0(\partial B_{|\xi|\varrho}^* (M)).$$

Our first result gives an ergodicity property of holomorphic continuations of ergodic eigenfunctions.

**Lemma 4.4.** Assume that $\{\varphi_{j\lambda}\}$ is a quantum ergodic sequence of $\Delta$-eigenfunctions on $M$ in the sense of (122). Then for each $0 < \epsilon < \epsilon_0$,  

$$|U_{j\lambda}|^2 \to \frac{1}{\mu_1(S^* M)}|\xi|^{-m+1}, \quad \text{weakly in } L^1(B_{\epsilon}^* M, \omega^m).$$

We note that $\omega^m = r^{m-1}dr d\omega d\text{vol}(x)$ in polar coordinates, so the right side indeed lies in $L^1$. The actual limit function is otherwise irrelevant. The next step is to use a compactness argument to obtain strong convergence of the normalized logarithms of the sequence $\{|U_{j\lambda}|^2\}$. The first statement of the following lemma immediately implies the second.

**Lemma 6.5.** Assume that $|U_{j\lambda}|^2 \to \frac{1}{\mu_1(S^* M)}|\xi|^{-m+1}$, weakly in $L^1(B_{\epsilon}^* M, \omega^m)$. Then:

1. $\frac{1}{\lambda_{j\lambda}} \log |U_{j\lambda}|^2 \to 0$ strongly in $L^1(B_{\epsilon}^* M)$.
2. $\frac{1}{\lambda_{j\lambda}} \partial \bar{\partial} \log |U_{j\lambda}|^2 \to 0$, weakly in $\mathcal{D}'(1,1)(B_{\epsilon}^* M)$.

Separating out the numerator and denominator of $|U_{j\lambda}|^2$, we obtain that

$$\frac{1}{\lambda_{j\lambda}} \partial \bar{\partial} \log |\varphi_{j\lambda}^{\epsilon}|^2 - \frac{2}{\lambda_{j\lambda}} \partial \bar{\partial} \log \rho_{j\lambda} \to 0, \quad (\lambda_{j\lambda} \to \infty).$$
The next lemma shows that the second term has a weak limit:

**Lemma 6.6.** For $0 < \epsilon < \epsilon_0$,
\[
\frac{1}{\lambda_{jk}} \log \rho_{\lambda_{jk}}(x, \xi) \to |\xi| g_\epsilon, \quad \text{in} \quad L^1(B_\epsilon^* M) \quad \text{as} \quad \lambda_{jk} \to \infty.
\]
Hence,
\[
\frac{1}{\lambda_{jk}} \partial \bar{\partial} \log \rho_{\lambda_{jk}} \to \partial \bar{\partial} |\xi| g_\epsilon, \quad (\lambda_j \to \infty) \quad \text{weakly in} \quad \mathcal{D}'(B_\epsilon^* M).
\]

It follows that the left side of (95) has the same limit, and that will complete the proof of Theorem B.1.

6.1. **Proof of Lemma 6.4.** We begin by proving a weak limit formula for the CR holomorphic functions $u^\epsilon_\lambda$ defined in (94) for fixed $\epsilon$. For notational simplicity, we drop the tilde notation although we work in the $B^*_\epsilon M$ setting.

**Lemma 6.7.** Assume that $\{\varphi_{jk}\}$ is a quantum ergodic sequence. Then for each $0 < \epsilon < \epsilon_0$,
\[
|u^\epsilon_{jk}|^2 \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)}, \quad \text{weakly in} \quad L^1(\partial B^*_\epsilon M, d\mu_\epsilon).
\]
That is, for any $a \in C(\partial B^*_\epsilon M)$,
\[
\int_{\partial B^*_\epsilon M} a(x, \xi)|u^\epsilon_{jk}|((x, \xi)|^2 d\mu_\epsilon \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a(x, \xi)d\mu_\epsilon.
\]

**Proof.** It suffices to consider $a \in C^\infty(\partial B^*_\epsilon M)$. We then consider the Toeplitz operator $\Pi_\epsilon a \Pi_\epsilon$ on $\mathcal{O}(\partial B^*_\epsilon M)$. We have,
\[
\langle \Pi_\epsilon a \Pi_\epsilon u^\epsilon_{jk}, u^\epsilon_{kl} \rangle = e^{2\alpha_j} \frac{\|\varphi^\epsilon_\lambda\|^2_{L^2(\partial B^*_\epsilon M)}}{\langle \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, U(i\epsilon)\varphi_j \rangle_{L^2(\partial B^*_\epsilon M)}}.
\]

It is not hard to see that $U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)$ is a pseudodifferential operator on $M$ of order $-\frac{m-1}{2}$ with principal symbol $\tilde{a} |\xi|^{m-2}$, where $\tilde{a}$ is the (degree 0) homogeneous extension of $a$ to $T^* M - 0$. The normalizing factor $e^{2\alpha_j} \frac{\|\varphi^\epsilon_\lambda\|^2_{L^2(\partial B^*_\epsilon M)}}{\langle \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, U(i\epsilon)\varphi_j \rangle_{L^2(\partial B^*_\epsilon M)}}$ has the same form with $a = 1$.

Hence, the expression on the right side of (96) may be written as
\[
\frac{\langle U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, U(i\epsilon)\varphi_j \rangle_{L^2(M)}}{\langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_j, U(i\epsilon)\varphi_j \rangle_{L^2(M)}},
\]

By the standard quantum ergodicity result on compact Riemannian manifolds with ergodic geodesic flow (see [Shn, Ze4, CV] for proofs and references) we have
\[
\frac{\langle U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}} \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a d\mu_\epsilon.
\]

More precisely, the numerator is asymptotic to the right side times $\lambda^{-\frac{m-1}{2}}$, while the denominator has the same asymptotics when $a$ is replaced by 1. We also use that $\frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a d\mu_\epsilon$ equals the analogous average of $\tilde{a}$ over $\partial B_1$ (see the discussion around (100)). Taking the ratio produces (98).
Combining (66), (68) and the fact that
\[ \langle \Pi, a \Pi \epsilon u_j^\epsilon, u_j^\epsilon \rangle = \int_{\partial B_2^M} a |u_j^\epsilon|^2 \, d\mu_\epsilon \]
completes the proof of the lemma.

We now complete the proof of Lemma 6.4, i.e. we prove that
\[ \int_{B_1^* M} a |U_j|^2 \omega^m \to \frac{1}{\mu_1(S^* M)} \int_{B_2^M} a |\xi|^{-m+1} \omega^m \]
for any \( a \in C(B_1^* M) \). It is only necessary to relate the Liouville measures \( d\mu_\epsilon \) to the symplectic volume measure. One may write \( d\mu_\epsilon = \frac{1}{\mu_1(S^* M)} \int_{\partial B_2^M} a |\xi|^{-m+1} \omega^m \) for any \( a \).

The second term on the right side is the matrix element of a pseudo-differential operator, hence is bounded by some power of \( \lambda \).

6.2. Proof of Lemma 6.6. In fact, one has
\[ \frac{1}{\lambda} \log \rho_\lambda(x, \xi) \to |\xi|_{g_\lambda}, \text{ uniformly in } B_2^* M \text{ as } \lambda \to \infty. \]

**Proof.** Again using \( U(\epsilon) \phi_\lambda = e^{-\lambda t} \phi_\lambda \), we have:
\[ \rho_\lambda^2(x, \xi) = \langle \Pi, e^{i \epsilon \pi} \Pi \phi_\lambda, \Pi \phi_\lambda \rangle_{L^2(M)} \quad (\epsilon = |\xi|_{g_\lambda}) \]

\[ = e^{2\lambda} \langle \Pi, U(\epsilon) \phi_\lambda, \Pi U(\epsilon) \phi_\lambda \rangle_{L^2(M)} \]

\[ = e^{2\lambda} \langle U(\epsilon)^* \Pi U(\epsilon) \phi_\lambda, \phi_\lambda \rangle_{L^2(M)}. \]

Hence,
\[ \frac{2}{\lambda} \log \rho_\lambda(x, \xi) = 2|\xi|_{g_\lambda} + \frac{1}{\lambda} \log \langle U^* \Pi U \phi_\lambda, \phi_\lambda \rangle. \]

The second term on the right side is the matrix element of a pseudo-differential operator, hence is bounded by some power of \( \lambda \). Taking the logarithm gives a remainder of order \( \frac{\log \lambda}{\lambda} \).

6.3. Proof of Lemma 6.5.

**Proof.** We wish to prove that
\[ \psi_j := \frac{1}{\lambda_j} \log |U_j|^2 \to 0 \text{ in } L^1(B_2^* M). \]
If the conclusion is not true, then there exists a subsequence $\psi_{j_k}$ satisfying $||\psi_{j_k}||_{L^1(B^*_\epsilon M)} \geq \delta > 0$. To obtain a contradiction, we use Lemma 4.1.

To see that the hypotheses are satisfied in our example, it suffices to prove these statements on each surface $\partial B^*_\epsilon M$ with uniform constants independent of $\epsilon$. On the surface $\partial B^*_\epsilon M$, $U_j = u_j^\epsilon$. By the Sobolev inequality in $O^{m-1}(\partial B^*_\epsilon M)$, we have

$$\sup_{(x,\xi) \in \partial B^*_\epsilon M} |u_j^\epsilon(x,\xi)| \leq \lambda_j^m|u_j^\epsilon(x)||L^2(\partial B^*_\epsilon M)$$

Taking the logarithm, dividing by $\lambda_j$, and combining with the limit formula of Lemma 6.6 proves (i) - (ii).

We now settle the dichotomy above by proving that the sequence $\{\psi_j\}$ does not tend uniformly to $-\infty$ on compact sets. That would imply that $\psi_j \to -\infty$ uniformly on the spheres $\partial B^*_\epsilon M$ for each $\epsilon < \epsilon_0$. Hence, for each $\epsilon$, there would exist $K > 0$ such that for $k \geq K$,

$$\frac{1}{\lambda_j} \log |u_j^\epsilon(z)| \leq -1. \tag{103}$$

However, (103) implies that

$$|u_j^\epsilon(z)| \leq e^{-2\lambda_j} \quad \forall z \in \partial B^*_e M,$$

which is inconsistent with the hypothesis that $|u_j^\epsilon(z)| \to 1$ in $D'(\partial B^*_\epsilon M)$.

Therefore, there must exist a subsequence, which we continue to denote by $\{\psi_{j_k}\}$, which converges in $L^1(B^{*}_e M)$ to some $\psi \in L^1(B^{*}_e M)$. Then,

$$\psi(z) = \limsup_{k \to \infty} \psi_{j_k} \leq 2|\xi|_g \quad (a.e.).$$

Now let

$$\psi^*(z) := \limsup_{w \to z} \psi(w) \leq 0$$

be the upper-semicontinuous regularization of $\psi$. Then $\psi^*$ is plurisubharmonic on $B^*_e M$ and $\psi^* = \psi$ almost everywhere.

If $\psi^* \leq 2|\xi|_g - \delta$ on a set $U_\delta$ of positive measure, then $\psi_{j_k}(\zeta) \leq -\delta/2$ for $\zeta \in U_\delta$, $k \geq K$; i.e.,

$$|\psi_{j_k}(\zeta)| \leq e^{-\delta}\lambda_j, \quad \zeta \in U_\delta, \quad k \geq K. \tag{104}$$

This contradicts the weak convergence to 1 and concludes the proof.

\[\square\]

7. INTERSECTIONS OF NODAL SETS AND ANALYTIC CURVES ON REAL ANALYTIC SURFACES

It is often possible to obtain more refined results on nodal sets by studying their intersections with some fixed (and often special) hypersurface. This has been most successful in dimension two. In §7.1 we discuss upper bounds on the number of intersection points of the nodal set with the boundary of a real analytic plane domain and more general ‘good’ analytic curves. To obtain lower bounds or asymptotics, we need to add some dynamical hypotheses. In case of ergodic geodesic flow, we can obtain equidistribution theorems for intersections of
nodal sets and geodesics on surfaces. The dimensional restriction is due to the fact that the
results are partly based on the quantum ergodic restriction theorems of \([TZ, TZ2]\), which
concern restrictions of eigenfunctions to hypersurfaces. Nodal sets and geodesics have com-
plementary dimensions and intersect in points, and therefore it makes sense to count the
number of intersections. But we do not yet have a mechanism for studying restrictions to
geodesics when \(\dim M \geq 3\).

7.1. Counting nodal lines which touch the boundary in analytic plane domains.

In this section, we review the results of \([TZ]\) giving upper bounds on the number of intersec-
tions of the nodal set with the boundary of an analytic (or more generally piecewise analytic)
plane domain. One may expect that the results of this section can also be generalized to
higher dimensions by measuring codimension two nodal hypersurface volumes within the
boundary.

Thus we would like to count the number of nodal lines (i.e. components of the nodal set)
which touch the boundary. Here we assume that 0 is a regular value so that components of
the nodal set are either loops in the interior (closed nodal loops) or curves which touch the
boundary in two points (open nodal lines). It is known that for generic piecewise analytic
plane domains, zero is a regular value of all the eigenfunctions \(\varphi_{\lambda_j}\), i.e. \(\nabla \varphi_{\lambda_j} \neq 0\) on
\(Z_{\varphi_{\lambda_j}} \cap \Omega\); we then call the nodal set regular. Since the boundary lies in the nodal set for
Dirichlet boundary conditions, we remove it from the nodal set before counting components.
Henceforth, the number of components of the nodal set in the Dirichlet case means the
number of components of \(Z_{\varphi_{\lambda_j}} \setminus \partial \Omega\).

We now sketch the proof of Theorems \([INTREALBdy]\) in the case of Neumann boundary conditions. By
a piecewise analytic domain \(\Omega^2 \subset \mathbb{R}^2\), we mean a compact domain with piecewise analytic
boundary, i.e. \(\partial \Omega\) is a union of a finite number of piecewise analytic curves which intersect
only at their common endpoints. Such domains are often studied as archetypes of domains
with ergodic billiards and quantum chaotic eigenfunctions, in particular the Bunimovich
stadium or Sinai billiard.

For the Neumann problem, the boundary nodal points are the same as the zeros of the
boundary values \(\varphi_{\lambda_j}|_{\partial \Omega}\) of the eigenfunctions. The number of boundary nodal points is thus
twice the number of open nodal lines. Hence in the Neumann case, the Theorem follows
from:

**Theorem 7.1.** Suppose that \(\Omega \subset \mathbb{R}^2\) is a piecewise real analytic plane domain. Then the
number \(n(\lambda_j) = \#Z_{\varphi_{\lambda_j}} \cap \partial \Omega\) of zeros of the boundary values \(\varphi_{\lambda_j}|_{\partial \Omega}\) of the \(j\)th Neumann
eigenfunction satisfies \(n(\lambda_j) \leq C_\Omega \lambda_j\), for some \(C_\Omega > 0\).

This is a more precise version of Theorem \([INTREALBdy]\) since it does not assume that 0 is a reg-
ular value. We prove Theorem 7.1 by analytically continuing the boundary values of the
eigenfunctions and counting complex zeros and critical points of analytic continuations of
Cauchy data of eigenfunctions. When \(\partial \Omega \in C^\omega\), the eigenfunctions can be holomorphically
continued to an open tube domain in \(\mathbb{C}^2\) projecting over an open neighborhood \(W\) in \(\mathbb{R}^2\) of
\(\Omega\) which is independent of the eigenvalue. We denote by \(\Omega_\zeta \subset \mathbb{C}^2\) the points \(\zeta = x + i\xi \in \mathbb{C}^2\)
with \(x \in \Omega\). Then \(\varphi_{\lambda_j}(x)\) extends to a holomorphic function \(\varphi_{\lambda_j}(\zeta)\) where \(x \in W\) and where
\(|\xi| \leq \epsilon_0\) for some \(\epsilon_0 > 0\).
Theorem 7.2. Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain, and denote by $(\partial \Omega)_C$ the union of the complexifications of its real analytic boundary components.

(1) Let $n(\lambda_j, \partial \Omega_C) = \# Z_{\varphi_{\lambda_j}}^{\partial \Omega_C}$ be the number of complex zeros on the complex boundary. Then there exists a constant $C_\Omega > 0$ independent of the radius of $(\partial \Omega)_C$ such that

$$n(\lambda_j, \partial \Omega_C) \leq C_\Omega \lambda_j.$$ 

The theorems on real nodal lines and critical points follow from the fact that real zeros and critical points are also complex zeros and critical points, hence

$$n(\lambda_j) \leq n(\lambda_j, \partial \Omega_C).$$

All of the results are sharp, and are already obtained for certain sequences of eigenfunctions on a disc (see [NIT]).

To prove 7.2, we represent the analytic continuations of the boundary values of the eigenfunctions in terms of layer potentials. Let $G(\lambda_j, x_1, x_2)$ be any ‘Green’s function’ for the Helmholtz equation on $\Omega$, i.e. a solution of $(\Delta - \lambda_j^2)G(\lambda_j, x_1, x_2) = \delta_{x_1}(x_2)$ with $x_1, x_2 \in \bar{\Omega}$. By Green’s formula,

$$\varphi_{\lambda_j}(x, y) = \int_{\partial \Omega} (\partial_\nu G(\lambda_j, q, (x, y)) \varphi_{\lambda_j}(q) - G(\lambda_j, q, (x, y)) \partial_\nu \varphi_{\lambda_j}(q)) \, d\sigma(q),$$

where $(x, y) \in \mathbb{R}^2$, where $d\sigma$ is arc-length measure on $\partial \Omega$ and where $\partial_\nu$ is the normal derivative by the interior unit normal. Our aim is to analytically continue this formula.

In the case of Neumann eigenfunctions $\varphi_{\lambda}$ in $\Omega$,

$$\varphi_{\lambda_j}(x, y) = \int_{\partial \Omega} \frac{\partial}{\partial v_q} G(\lambda_j, q, (x, y)) u_{\lambda_j}(q) d\sigma(q), \quad (x, y) \in \Omega^o \quad \text{(Neumann).}$$

To obtain concrete representations we need to choose $G$. We choose the real ambient Euclidean Green’s function $S$

$$S(\lambda_j, \xi, \eta; x, y) = -Y_0(\lambda_j r((x, y); (\xi, \eta))),$$

where $r = \sqrt{zz^*}$ is the distance function (the square root of $r^2$ above) and where $Y_0$ is the Bessel function of order zero of the second kind. The Euclidean Green’s function has the form

$$S(\lambda_j, \xi, \eta; x, y) = A(\lambda_j, \xi, \eta; x, y) \log \frac{1}{r} + B(\lambda_j, \xi, \eta; x, y),$$

where $A$ and $B$ are entire functions of $r^2$. The coefficient $A = J_0(\lambda_j r)$ is known as the Riemann function.

By the ‘jumps’ formulae, the double layer potential $\frac{\partial}{\partial v_q} S(\lambda_j, q, (x, y))$ on $\partial \Omega \times \bar{\Omega}$ restricts to $\partial \Omega \times \partial \Omega$ as $\frac{1}{2} \delta_q(q) + \frac{\partial}{\partial v_q} S(\lambda_j, q, q)$ (see e.g. [II, TII]). Hence in the Neumann case the boundary values $u_{\lambda_j}$ satisfy,

$$u_{\lambda_j}(q) = 2 \int_{\partial \Omega} \frac{\partial}{\partial v_q} S(\lambda_j, q, q) u_{\lambda_j}(q) d\sigma(q) \quad \text{(Neumann).}$$
We have,

\[
\frac{\partial}{\partial \nu_q} S(\lambda_j, \tilde{q}, q) = -\lambda_j Y_1(\lambda_j r) \cos \angle(q - \tilde{q}, \nu_q).
\]

It is equivalent, and sometimes more convenient, to use the (complex valued) Euclidean outgoing Green’s function \( H^{(1)}_a(kz) \), where \( H^{(1)}_a = J_0 + iY_0 \) is the Hankel function of order zero. It has the same form as \( (109) \) and only differs by the addition of the even entire function \( J_0 \) to the \( B \) term. If we use the Hankel free outgoing Green’s function, then in place of \( (111) \) we have the kernel

\[
N(\lambda_j, q(s), q(s')) = \frac{i}{2} \partial_{\nu_q} H^{(1)}_a(\lambda_j |q(s) - y|) |_{y=q(s')}
\]

and in place of \( (110) \) we have the formula

\[
\int_{\lambda_j}^{\text{Hankel}} u_{\lambda_j}(q(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q(t)) u_{\lambda_j}(q(s)) ds.
\]

The next step is to analytically continue the layer potential representations \( (110) \) and \( (113) \). The main point is to express the analytic continuations of Cauchy data of Neumann and Dirichlet eigenfunctions in terms of the real Cauchy data. For brevity, we only consider \( (110) \) but essentially the same arguments apply to the free outgoing representation \( (113) \).

As mentioned above, both \( A(\lambda_j, \xi, \eta, x, y) \) and \( B(\lambda_j, \xi, \eta, x, y) \) admit analytic continuations. In the case of \( A \), we use a traditional notation \( R(\zeta, \zeta^*, z, z^*) \) for the analytic continuation and for simplicity of notation we omit the dependence on \( \lambda_j \).

The details of the analytic continuation are complicated when the curve is the boundary, and they simplify when the curve is interior. So we only continue the sketch of the proof in the interior case.

As above, the arc-length parametrization of \( C \) is denoted by \( q_C : [0, 2\pi] \to C \) and the corresponding arc-length parametrization of the boundary, \( \partial C \), by \( q : [0, 2\pi] \to \partial C \). Since the boundary and \( C \) do not intersect, the logarithm \( \log r^2(q(s); q_C^*(t)) \) is well defined and the holomorphic continuation of equation \( (113) \) is given by:

\[
\int_{T} \phi_{\lambda_j}^{\text{Hankel}}(q_C^*(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q_C^*(t)) u_{\lambda_j}(q(s)) ds,
\]

From the basic formula \( (112) \) for \( N(\lambda_j, q, q_C) \) and the standard integral formula for the Hankel function \( H^{(1)}_a(z) \), one easily gets an asymptotic expansion in \( \lambda_j \) of the form:

\[
N(\lambda_j, q(s), q_C^*(t)) = e^{i\lambda_j r(q(s); q_C^*(t))} \sum_{m=0}^{k} a_m(q(s), q_C^*(t)) \lambda_j^{1/2-m} + O(e^{i\lambda_j r(q(s); q_C^*(t))} \lambda_j^{1/2-k-1})..
\]

Note that the expansion in \( (115) \) is valid since for interior curves,

\[
C_0 := \min_{(q_C(t), q(s)) \in C \times \partial C} |q_C(t) - q(s)|^2 > 0.
\]
Then, \( \text{Re} r^2(q(s); q_C^C(t)) > 0 \) as long as
\[
\| \text{Im} q_C^C(t) \|^2 < C_0. \tag{116}
\]
So, the principal square root of \( r^2 \) has a well-defined holomorphic extension to the tube (116) containing \( C \). We have denoted this square root by \( r \) in (115).

Substituting (115) in the analytically continued single layer potential integral formula (114) proves that for \( t \in A(\epsilon) \) and \( \lambda_j > 0 \) sufficiently large,
\[
\varphi_{\lambda_j}^C(q_C^C(t)) = 2\pi \lambda_j^{1/2} \int_0^{2\pi} e^{i\lambda_j r(q(s); q_C^C(t))} a_0(q(s), q_C^C(t))(1 + O(\lambda_j^{-1})) u_{\lambda_j}(q(s)) d\sigma(s). \tag{117}
\]
Taking absolute values of the integral on the RHS in (117) and applying the Cauchy-Schwartz inequality proves

**Lemma 7.3.** For \( t \in [0, 2\pi] + i[-\epsilon, \epsilon] \) and \( \lambda_j > 0 \) sufficiently large
\[
|\varphi_{\lambda_j}^C(q_C^C(t))| \leq C_1 \lambda_j^{1/2} \exp \lambda_j \left( \max_{q(s) \in \partial \Omega} \text{Re} \, ir(q(s); q_C^C(t)) \right) \cdot \| u_{\lambda_j} \|_{L^2(\partial \Omega)}.
\]

From the pointwise upper bounds in Lemma 7.3, it is immediate that
\[
\log \max_{q_C^C(t) \in Q_C^C(A(\epsilon))} |\varphi_{\lambda_j}^C(q_C^C(t))| \leq C_{\text{max}} \lambda_j + C_2 \log \lambda_j + \log \| u_{\lambda_j} \|_{L^2(\partial \Omega)}, \tag{118}
\]
where,
\[
C_{\text{max}} = \max_{(q(s), q_C^C(t)) \in \partial \Omega \times Q_C^C(A(\epsilon))} \text{Re} \, ir(q(s); q_C^C(t)).
\]

Finally, we use that \( \log \| u_{\lambda_j} \|_{L^2(\partial \Omega)} = O(\lambda_j) \) by the assumption that \( C \) is a good curve and apply Proposition 7.5 to get that \( n(\lambda_j, C) = O(\lambda_j) \).

The following estimate, suggested by Lemma 6.1 of Donnelly-Fefferman [DF], gives an upper bound on the number of zeros in terms of the growth of the family:

**Proposition 7.4.** Suppose that \( C \) is a good real analytic curve in the sense of (24). Normalize \( u_{\lambda_j} \) so that \( \| u_{\lambda_j} \|_{L^2(C)} = 1 \). Then, there exists a constant \( C(\epsilon) > 0 \) such that for any \( \epsilon > 0 \),
\[
n(\lambda_j, Q_C^C(A(\epsilon/2))) \leq C(\epsilon) \max_{q_C^C(t) \in Q_C^C(A(\epsilon))} \log |u_{\lambda_j}^C(q_C^C(t))|.
\]

**Proof.** Let \( G_\epsilon \) denote the Dirichlet Green’s function of the ‘annulus’ \( Q_C^C(A(\epsilon)) \). Also, let \( \{a_k\}_{k=1}^{n(\lambda_j, Q_C^C(A(\epsilon/2)))} \) denote the zeros of \( u_{\lambda_j}^C \) in the sub-annulus \( Q_C^C(A(\epsilon/2)) \). Let \( U_{\lambda_j} = \frac{u_{\lambda_j}^C}{\| u_{\lambda_j}^C \|_{Q_C^C(A(\epsilon))}} \) where \( \| u \|_{Q_C^C(A(\epsilon))} = \max_{\zeta \in Q_C^C(A(\epsilon))} |u(\zeta)|. \) Then,
\[
\log |U_{\lambda_j}(q_C^C(t))| = \int_{Q_C^C((A(\epsilon/2)))} G_\epsilon(q_C^C(t), w) \partial \partial \log |u_{\lambda_j}(w)| + H_{\lambda_j}(q_C^C(t))
\]
\[
= \sum_{a_k \in Q_C^C(A(\epsilon/2))} G_\epsilon(q_C^C(t), a_k) + H_{\lambda_j}(q_C^C(t)),
\]
since \( \partial \partial \log |u_{\lambda_j}(w)| = \sum_{a_k \in C_C: u_{\lambda_j}(a_k) = 0} \delta_{a_k} \). Moreover, the function \( H_{\lambda_j} \) is sub-harmonic on \( Q_C^C(A(\epsilon)) \) since
\[
\partial \partial H_{\lambda_j} = \partial \partial \log |U_{\lambda_j}(q_C^C(t))| - \sum_{a_k \in Q_C^C(A(\epsilon/2)): u_{\lambda_j}(a_k) = 0} \partial \partial G_\epsilon(q_C^C(t), a_k)
\]
So, by the maximum principle for subharmonic functions,
\[
\max_{Q_C^C(A(\epsilon))} H_{\lambda_j}(q_C^C(t)) \leq \max_{\partial Q_C^C(A(\epsilon))} H_{\lambda_j}(q_C^C(t)) = \max_{\partial Q_C^C(A(\epsilon))} \log |U_{\lambda_j}(q_C^C(t))| = 0.
\]
It follows that
\[
(119) \quad \log |U_{\lambda_j}(q_C^C(t))| \leq \sum_{a_k \in Q_C^C(A(\epsilon/2))} G_\epsilon(q_C^C(t), a_k),
\]
hence that
\[
(120) \quad \max_{q_C^C(t) \in Q_C^C(A(\epsilon/2))} \log |U_{\lambda_j}(q_C^C(t))| \leq \left( \max_{z,w \in Q_C^C(A(\epsilon/2))} G_\epsilon(z,w) \right) n(\lambda_j, Q_C^C(A(\epsilon/2))).
\]
Now \( G_\epsilon(z,w) \leq \max_{w \in Q_C^C(\partial A(\epsilon))} G_\epsilon(z,w) = 0 \) and \( G_\epsilon(z,w) < 0 \) for \( z,w \in Q_C^C(A(\epsilon/2)) \). It follows that there exists a constant \( \nu(\epsilon) < 0 \) so that \( \max_{z,w \in Q_C^C(A(\epsilon/2))} G_\epsilon(z,w) \leq \nu(\epsilon) \). Hence,
\[
(121) \quad \max_{q_C^C(t) \in Q_C^C(A(\epsilon/2))} \log |U_{\lambda_j}(q_C^C(t))| \leq \nu(\epsilon) \ n(\lambda_j, Q_C^C(A(\epsilon/2))).
\]
Since both sides are negative, we obtain
\[
(122) \quad \leq \frac{1}{|\nu(\epsilon)|} \left( \max_{q_C^C(t) \in Q_C^C(A(\epsilon))} \log |u_{\lambda_j}^C(q_C^C(t))| - \max_{q_C^C(t) \in Q_C^C(A(\epsilon/2))} \log |u_{\lambda_j}^C(q_C^C(t))| \right)
\]
\[
\leq \frac{1}{|\nu(\epsilon)|} \max_{q_C^C(t) \in Q_C^C(A(\epsilon))} \log |u_{\lambda_j}^C(q_C^C(t))|,
\]
where in the last step we use that \( \max_{q_C^C(t) \in Q_C^C(A(\epsilon/2))} \log |u_{\lambda_j}^C(q_C^C(t))| \geq 0 \), which holds since \( |u_{\lambda_j}^C| \geq 1 \) at some point in \( Q_C^C(A(\epsilon/2)) \). Indeed, by our normalization, \( \|u_{\lambda_j}\|_{L^2(C)} = 1 \), and so there must already exist points on the real curve \( C \) with \( |u_{\lambda_j}| \geq 1 \). Putting \( C(\epsilon) = \frac{1}{|\nu(\epsilon)|} \) finishes the proof.

This completes the proof of Theorem \( \text{GOODTH} \).

7.2. Application to Pleijel’s conjecture. I. Polterovich [Po] observed that Theorem 6 can be used to prove an old conjecture of A. Pleijel regarding Courant’s nodal domain theorem, which says that the number \( n_k \) of nodal domains (components of \( \Omega \setminus Z_{\varphi_\lambda_k} \)) of the \( k \)th eigenfunction satisfies \( n_k \leq k \). Pleijel improved this result for Dirichlet eigenfunctions of plane domains: For any plane domain with Dirichlet boundary conditions, \( \limsup_{k \to \infty} \frac{n_k}{k} \leq \frac{4}{j_1} \approx 0.691... \), where \( j_1 \) is the first zero of the \( J_0 \) Bessel function. He conjectured that the same result should be true for a free membrane, i.e. for Neumann boundary conditions. This was recently proved in the real analytic case by I. Polterovich [Po]. His argument is roughly the following: Pleijel’s original argument applies to all nodal domains which do not touch the boundary, since the eigenfunction is a Dirichlet eigenfunction in such a nodal domain. The argument does not apply to nodal domains which touch the boundary, but by the Theorem above the number of such domains is negligible for the Pleijel bound.
7.3. Equidistribution of intersections of nodal lines and geodesics on surfaces. We fix $(x, \xi) \in S^*M$ and let
\begin{equation}
\gamma_{x, \xi} : \mathbb{R} \to M, \quad \gamma_{x, \xi}(0) = x, \quad \gamma'_{x, \xi}(0) = \xi \in T_x M
\end{equation}
denote the corresponding parametrized geodesic. Our goal is to determine the asymptotic distribution of intersection points of $\gamma_{x, \xi}$ with the nodal set of a highly eigenfunction. As usual, we cannot cope with this problem in the real domain and therefore analytically continue it to the complex domain. Thus, we consider the intersections
\begin{equation}
N_{\lambda_j}^{C, x, \xi} = Z_{\phi_j^C} \cap \gamma_{x, \xi}^C
\end{equation}
of the complex nodal set with the (image of the) complexification of a generic geodesic. If
\begin{equation}
S_\epsilon = \{(t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\}
\end{equation}
then $\gamma_{x, \xi}$ admits an analytic continuation
\begin{equation}
\gamma_{x, \xi}^C : S_\epsilon \to M_\epsilon.
\end{equation}
In other words, we consider the zeros of the pullback,
\begin{equation}
\{\gamma^*_{x, \xi} \phi^C_\lambda = 0\} \subset S_\epsilon.
\end{equation}

We encode the discrete set by the measure
\begin{equation}
[N_{\lambda_j}^{C, x, \xi}] = \sum_{(t+i\tau) : \phi_j^C(\gamma_{x, \xi}^C(t+i\tau)) = 0} \delta_{t+i\tau}.
\end{equation}

We would like to show that for generic geodesics, the complex zeros on the complexified geodesic condense on the real points and become uniformly distributed with respect to arc-length. This does not always occur: as in our discussion of QER theorems, if $\gamma_{x, \xi}$ is the fixed point set of an isometric involution, then “odd” eigenfunctions under the involution will vanish on the geodesic. The additional hypothesis is that QER holds for $\gamma_{x, \xi}$. The following is proved ([Ze3]):

**THEOREM 7.5.** Let $(M^2, g)$ be a real analytic Riemannian surface with ergodic geodesic flow. Let $\gamma_{x, \xi}$ satisfy the QER hypothesis. Then there exists a subsequence of eigenvalues $\lambda_{jk}$ of density one such that for any $f \in C_c(S_\epsilon)$,
\begin{equation}
\lim_{k \to \infty} \sum_{(t+i\tau) : \phi_j^C(\gamma_{x, \xi}^C(t+i\tau)) = 0} f(t+i\tau) = \int_{\mathbb{R}} f(t) dt.
\end{equation}

In other words,
\begin{equation}
\text{weak}^* \lim_{k \to \infty} \frac{i}{\pi \lambda_{jk}} [N_{\lambda_j}^{C, x, \xi}] = \delta_{\tau = 0},
\end{equation}
in the sense of weak* convergence on $C_c(S_\epsilon)$. Thus, the complex nodal set intersects the (parametrized) complexified geodesic in a discrete set which is asymptotically (as $\lambda \to \infty$) concentrated along the real geodesic with respect to its arclength.

This concentration-equidistribution result is a ‘restricted’ version of the result of §6. As noted there, the limit distribution of complex nodal sets in the ergodic case is a singular current $dd^c \sqrt{\rho}$. The motivation for restricting to geodesics is that restriction magnifies the
singularity of this current. In the case of a geodesic, the singularity is magnified to a delta-function; for other curves there is additionally a smooth background measure.

The assumption of ergodicity is crucial. For instance, in the case of a flat torus, say \( \mathbb{R}^2/L \) where \( L \subset \mathbb{R}^2 \) is a generic lattice, the real eigenfunctions are \( \cos(\lambda, x), \sin(\lambda, x) \) where \( \lambda \in L^* \), the dual lattice, with eigenvalue \(-|\lambda|^2\). Consider a geodesic \( \gamma_{x,\xi}(t) = x + t\xi \). Due to the flatness, the restriction \( \sin(\lambda, x_0 + t\xi_0) \) of the eigenfunction to a geodesic is an eigenfunction of the Laplacian \(-\frac{d^2}{dt^2}\) of submanifold metric along the geodesic with eigenvalue \(-\langle \lambda, \xi_0 \rangle^2\).

The complexification of the restricted eigenfunction is \( \sin \) of its growth is \( \tau \). Thus, the main point of the proof is to determine the asymptotics of \( N \) many limits along different rays in \( L \) where \( L \) function; for other curves there is additionally a smooth background measure. In the case of a geodesic, the singularity is magnified to a delta-sequence \( 1 \) in \( L \). Hence the difficult point is to prove that this growth rate is actually obtained when we freeze \( \tau \) over the intersection points in \((\lambda, \xi_0)\) and which are specific to geodesics. However, the first steps in the proof are the same as in the flatness, the restriction \( \sin \) of the Laplacian -\(|\lambda|^2\) of submanifold metric along the geodesic with eigenvalue \(-\langle \lambda, \xi_0 \rangle^2\). The quantum ergodic restriction theorem in the real domain shows that the Fourier modes, i.e. higher modes are exponentially larger than lower modes.

Complexifications of restrictions of eigenfunctions to geodesics have incommensurate \( \tau \); tend to constant multiples of Lebesgue measures \( dt \) for each \( \tau > 0 \). Hence the eigenfunctions everywhere on \( \gamma_{x,\xi}^C \) achieve the growth rate of the \( L^2 \) norms.
These principles are most easily understood in the case of periodic geodesics. We let 
\[ \gamma_{x,\xi} : S^1 \to M \] parametrize the geodesic with arc-length (where \( S^1 = \mathbb{R}/L\mathbb{Z} \) where \( L \) is the length of \( \gamma_{x,\xi} \)).

**Lemma 7.7.** Assume that \( \{ \varphi_j \} \) satisfies QER along the periodic geodesic \( \gamma_{x,\xi} \). Let \( ||\gamma_{x,\xi}^{\tau^*}\varphi_j^c||^2_{L^2(S^1)} \) be the \( L^2 \)-norm of the complexified restriction of \( \varphi_j \) along \( \gamma_{x,\xi}^{\tau} \). Then,

\[
\lim_{\lambda_j \to \infty} \frac{1}{\lambda_j} \log ||\gamma_{x,\xi}^{\tau^*}\varphi_j^c||^2_{L^2(S^1)} = |\tau|.
\]

To prove Lemma 7.7, we study the orbital Fourier series of \( \gamma_{x,\xi}^{\tau^*}\varphi_j \) and of its complexification. The orbital Fourier coefficients are

\[
\nu_{\lambda_j}^{x,\xi}(n) = \frac{1}{L} \int_0^{L\gamma} \varphi_{\lambda_j}(\gamma_{x,\xi}(t)) e^{-2\pi i nt} dt,
\]

and the orbital Fourier series is

\[
\varphi_{\lambda_j}(\gamma_{x,\xi}(t)) = \sum_{n \in \mathbb{Z}} \nu_{\lambda_j}^{x,\xi}(n) e^{2\pi i nt}.
\]

Hence the analytic continuation of \( \gamma_{x,\xi}^{\tau^*}\varphi_j \) is given by

\[
\varphi_{\lambda_j}^c(\gamma_{x,\xi}(t + i \tau)) = \sum_{n \in \mathbb{Z}} \nu_{\lambda_j}^{x,\xi}(n) e^{2\pi i (t+n\tau)}.
\]

By the Paley-Wiener theorem for Fourier series, the series converges absolutely and uniformly for \( |\tau| \leq \epsilon_0 \). By “energy localization” only the modes with \( |n| \leq \lambda_j \) contribute substantially to the \( L^2 \) norm. We then observe that the Fourier modes decouple, since they have different exponential growth rates. We use the QER hypothesis in the following way:

**Lemma 7.8.** Suppose that \( \{ \varphi_{\lambda_j} \} \) is QER along the periodic geodesic \( \gamma_{x,\xi} \). Then for all \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) so that

\[
\sum_{n : |n| \geq (1-\epsilon)\lambda_j} \left| \nu_{\lambda_j}^{x,\xi}(n) \right|^2 \geq C_\epsilon.
\]

**Lemma 7.8** implies Lemma 7.7 since it implies that for any \( \epsilon > 0 \),

\[
\sum_{n : |n| \geq (1-\epsilon)\lambda_j} \left| \nu_{\lambda_j}^{x,\xi}(n) \right|^2 e^{-2\pi n \tau} \geq C_\epsilon e^{2\pi (1-\epsilon)\lambda_j}.
\]

To go from asymptotics of \( L^2 \) norms of restrictions to Proposition 7.6 we then use the third principle:

**Proposition 7.9.** (Lebesgue limits) If \( \gamma_{x,\xi}^*\varphi_j \neq 0 \) (identically), then for all \( \tau > 0 \) the sequence

\[
U_{x,\xi,\tau} = \frac{\gamma_{x,\xi}^{\tau^*}\varphi_j^c}{||\gamma_{x,\xi}^{\tau^*}\varphi_j^c||_{L^2(S^1)}}
\]

is QUE with limit measure given by normalized Lebesgue measure on \( S^1 \).

The proof of Proposition 7.6 is completed by combining Lemma 7.7 and Proposition 7.9. Theorem 7.5 follows easily from Proposition 7.6.

The proof for non-periodic geodesics is considerably more involved, since one cannot use Fourier analysis in quite the same way.
7.4. Real zeros and complex analysis.

**Problem 4.** An important but apparently rather intractable problem is, how to obtain information on the real zeros from knowledge of the complex nodal distribution? There are several possible approaches:

- Try to intersect the nodal current with the current of integration over the real points \( M \subset M_c \). I.e. try to slice the complex nodal set with the real domain.

- Thicken the real slice slightly by studying the behavior of the nodal set in \( M_\epsilon \) as \( \epsilon \to 0 \). The sharpest version is to try to re-scale the nodal set by a factor of \( \lambda^{-1} \) to zoom in on the zeros which are within \( \lambda^{-1} \) of the real domain. They may not be real but at least one can control such “almost real” zeros. Try to understand (at least in real dimension 2) how the complex nodal set ‘sprouts’ from the real nodal set. How do the connected components of the real nodal set fit together in the complex nodal set?

- Intersect the nodal set with geodesics. This magnifies the singularity along the real domain and converts nodal sets to isolated points.

8. Complex nodal sets in the completely integrable case

As mentioned in the introduction, we consider zeros of (holomorphic extensions of)

\[
\text{Re} \varphi_\alpha(x) = \varphi_\alpha(x) + \varphi_{-\alpha}(x)
\]

Thus we consider the sequence of zero sets,

\[
\mathcal{N}_{k\alpha} = \{ \zeta \in M_C : \varphi^\mathcal{C}_{k\alpha}(\zeta) + \varphi^\mathcal{C}_{-k\alpha}(\zeta) = 0 \},
\]

where \( \alpha \) is a lattice point and \( k = 1, 2, \ldots \). Thus the limit is along rays in the joint spectrum.

The terms \( \varphi^\mathcal{C}_{\pm k\alpha}(\zeta) \) have “opposite regions” of exponential growth/decay. They only cancel out along the anti-Stokes hypersurface where they have the same size.

The formula for the limits of delta-functions on nodal sets involves complex travel times for the Hamiltonian torus action

\[
\Phi_{t}(z), \quad z \in M_c, \quad \vec{t} \in T^{m}, \quad m = \dim M.
\]

The orbits \( \vec{t} \to \Phi_{t}(z) \), are the level sets of the moment map \( \mathcal{P} \). When \( z \notin \mathcal{P}^{-1}(\alpha) \), then the size \( |\varphi_{k\alpha}(z)| \) involves the distance of \( z \) from \( \mathcal{P}^{-1}(\alpha) \)– measured by \( \tau(z, \alpha) \in \mathbb{R}^n \) so that

\[
\Phi_{t\tau}(z) \in \mathcal{P}^{-1}(\alpha).
\]

The imaginary time orbit “crosses levels of the moment map”, like a joint gradient flow. \( \tau(z, \alpha) \) is the “tunnelling time.”
As stated in Theorem 12 and Theorem 13 the limit current of the nodal sets

\[ N_{k\alpha} := \{ z : \varphi_{k\alpha}(z) + \varphi_{k,-\alpha}(z) = 0 \} \]

along a ray \( k\alpha : k = 1, 2, \ldots \) in the joint spectrum forms a real hypersurface (the ‘anti-Stokes surface’) in \( M_c \). The exponential growth rate of the complexified eigenfunction is \( e^{kG} \) where

\[ G_\alpha = 2 \left\| \langle \vec{t}(z, \alpha), H(\alpha) \rangle + \sqrt{\rho}(z) \right\|, \]

where \( \tau(z, \alpha) \) is the imaginary travel time from \( z \) to \( \mathcal{I}^{-1}(\alpha) \). \( \tau(z, \alpha) \) measures how far \( z \) is from the level set \( P = \alpha \) of the moment map in terms of complexified travel time. The anti-Stokes set is \( G_\alpha = 0 \).

8.1. **Reduction to growth rates along ladders.** As in the ergodic case, one can reduce the problem of finding limit currents of integration over nodal sets to finding the exponential growth rates along ladders:

**Proposition 8.1.** Let \( N_{\alpha}^C \) be the complex nodal set for \( \text{Re} \varphi_\alpha \). Then for a ladder \( L_\alpha = \{ k\alpha, k \in \mathbb{Z}_+ \} \), \( u_\alpha \) is well-defined and the limit distribution of the nodal set currents along the ladder is given by

\[
\lim_{k \to \infty} \frac{1}{kH(\alpha)} \int_{N_{k\alpha}^C} f_{\omega} m^{-1} \to i \int_{M_c} f_{\omega} m^{-1} \wedge \partial \bar{\partial} u_\alpha.
\]

The same limit formula applies to the complex eigenfunctions \( \varphi_{k\alpha} \) and shows that they are zero free if and only if \( \sqrt{\rho} \) is harmonic, as is easily seen to be the case on a flat torus (where they are linear). The limit formula of Proposition 9.12 reduces the equidistribution theory of complex nodal sets to the question of determining \( u_\alpha \).

Thus, the main problem is to determine exponential growth rates of complexified eigenfunctions \( \varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y) \) along ladders. For this it suffices to determine the exponential growth rates of the normalized eigenfunctions, \( \frac{\varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y)}{||\varphi_{k\alpha}^C||_2} \). Here, \( ||\varphi_{k\alpha}^C||_2 \) is the \( L^2 \) norm of \( \varphi_{k\alpha}^C \) on \( \partial M_c \) with respect to the natural (Liouville) surface measure. The norm in the denominator is harmless because its asymptotics are easily found in

**Proposition 8.2.** Let \( \{ \varphi_{k\alpha} \} \) be a ladder of \( L^2 \)-normalized joint eigenfunctions. Then we have,

\[
\frac{1}{kH(\alpha)} \log ||\varphi_{k\alpha}^C||_{L^2(\partial M_c)} \to |\epsilon|.
\]

8.2. **Stationary phase and oscillatory integrals.** Although Theorem 12 only requires knowledge of the exponential growth rates of complexified eigenfunctions, we prove it by finding their stationary phase asymptotics:

**Theorem 8.3.** Let \( (z, y) \in \partial M_c \times \partial M_c \), and let \( (\vec{t} + i\vec{\tau})(z, y) \) be the complex travel time of Definition 11. Then there exists a semi-classical symbol \( \mathcal{A}_0 \) of order \( m \) so that

\[
\frac{\varphi_{k\alpha}^C(z) \varphi_{k\alpha}^C(y)}{||\varphi_{k\alpha}^C||_2^2} \simeq \mathcal{A}_0(k, z, y) e^{ik((\tilde{t}_0 + i\tau(z, \alpha, y)))}. \]
To prove Theorem \ref{thm:131}, we construct special oscillatory integral formulae for eigenfunctions:

\[
\frac{\varphi^C_k(\alpha \varphi^C_{k\alpha}(y))}{\|\varphi^C_k\|^2} = \int_0^\infty \int_0^\infty \int_{\partial M_\epsilon} A_k(z, w, \theta_1, \vec{t}, w, y, \theta_2) d\mu_\epsilon(w) d\theta_1 d\theta_2.
\]

This is based on constructing the Fourier integral torus action as a Toeplitz dynamical operator as in Proposition \ref{prop:5.3}, but with \(\tilde{\Phi}_t\) in place of \(g^t\). We then take the Fourier coefficient corresponding to the character \(e^{i\langle \alpha, \vec{t} \rangle}\).

The phase is the function on \(T^m \times \partial M_\epsilon \times \mathbb{R}_+ \times \mathbb{R}_+\) defined by

\[
\Psi_\alpha(\vec{t}, w, \theta_1, \theta_2; z, y) := \theta_1 \psi(z, w) + \theta_2 \psi(\tilde{\Phi}_t w, y) - \langle \alpha, \vec{t} \rangle,
\]

where

\[
\psi(z, w) = \frac{1}{i}(\rho(z, w) - 4e^2), \quad (z, w) \in \partial M_\epsilon.
\]

Also \(\tilde{\Phi}_t\) is the torus action transported to \(\partial M_\epsilon\) by the complexified exponential map \(E : B^*_M \to \partial M_\epsilon\).

To prove Theorem \ref{thm:8.3} we apply the saddle point method in the complex domain. The crucial point is that the growth rate in Theorem \ref{thm:12} is given by the value of the phase function (\ref{eq:133}) on the critical set.

In analyzing the critical point equations, it is assumed that \(z, y \in \partial M_\epsilon\). We are especially interested in the cases:

1. \(y = z\);
2. \(y = z_\alpha \in E(\Lambda_\alpha)\).

The critical point equations in \((s, w, \vec{t}, \theta_1, \theta_2)\) are

\[
\left\{
\begin{aligned}
(i) & \quad \psi(z, w) = 0, \quad \psi(\tilde{\Phi}_t(w), y) = 0 \\
(ii) & \quad \theta_1 d_w \rho(z, w) + \theta_2 d_w \rho(\tilde{\Phi}_t(w), y) = 0, \\
(iii) & \quad \theta_2 \nabla_{\vec{t}} \rho(\tilde{\Phi}_t(w), y) = \alpha.
\end{aligned}
\right.
\]

Remark:

It is obvious that the equation has no real solution \(\vec{t} \in T^m\) when \(z, y\) lie on different \(T^m\) orbits. We therefore have to let \(\vec{t} \in T^c_m\). But then equation (iii) changes to a deformed moment map equation and it is not possible that \(\mathcal{I}(y) = \mathcal{I}(^t)(y) = \alpha\) for all (or even an open set of) \(\vec{t}\). One might think that \(\nabla_{\vec{t}} \rho(\tilde{\Phi}^{t+i\vec{t}}_t W, y) = \alpha\), and this would hold if \(\tilde{\Phi}^{t+i\vec{t}}_t W = \tilde{\Phi}^t_\vec{t} W\) and if \(\tilde{\Phi}^{t+i\vec{t}}_t W = y\). But the group property does not hold when \(\tilde{\Phi}^t_\vec{t}\) fails to be a holomorphic group action.

It follows that there are no solutions \(w, \vec{t} + i\vec{t}\) of the critical points equations with \(w\) and \(\tilde{\Phi}^{t+i\vec{t}}_t W\) in the “real domain” \(\partial M_\epsilon\), unless \(z, y \in \Lambda_\alpha^\epsilon\). Indeed, the only solution of (\ref{eq:133}) (1) in the “real domain” \(\partial M_\epsilon\) occurs when \(w = z\) and \(\tilde{\Phi}^t_\vec{t}(z) = y\).
The main difficulty is that unless \( z \in \tilde{\Lambda}_a \), the phase \( \Phi \) has no critical points along the contour of integration or even in \( \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{T}_C^m \times \partial M_e \). Indeed, \( \nabla y \rho(\tilde{\Phi} + i\tau) (z, z) \neq \alpha \) in general when \( \tilde{\Phi}(\tilde{\Phi} + i\tau)(z) = z \). As discussed above, we therefore need to deform the contour into the complexification \( \mathbb{T}_C^m \times (\partial M_e)_{C^2} \), where \( \partial M_e \) is regarded as a real manifold.

We therefore analytically continue \( \Phi \) to \( \mathbb{T}_C^m \times (M_e)_{C^2} \) as a holomorphic symplectic group action satisfying \( \tilde{\Phi}_{t + i\tau} W = \tilde{\Phi}_t \tilde{\Phi}_{i\tau} W \). The problem then is to find \( W \in (\partial M_e)_{C^2} \) so that \( (\Phi_t + i\tau) W, y = \alpha \). Since \( y \in \partial M_e \) and \( W \notin \partial M_e \), we must work off the diagonal with \( \rho(z, W) \).

A useful fact about complexifying a complex manifold \( Y \) is that \( Y_{\mathbb{C}} \cong Y \times \tilde{Y} \) where \( \tilde{Y} \) is the complex conjugate manifold. We then regard the original manifold \( Y \) as the totally real anti-diagonal submanifold \( \{(y, \bar{y}) \} \subset Y \times \tilde{Y} \). We use the notation \( (Y', Y'') \) for points of \( Y \times \tilde{Y} \), so that the anti-diagonal is the set where \( Y'' = \bar{Y}' \).

The main result of the stationary phase analysis is the following

**Proposition 8.4.** For \( z \in \partial M_e \) and \( y = z_a \in \Lambda_a \), there exists a non-degenerate critical point

\[(t, W, \theta_1, \theta_2) \in \mathbb{T}_C^m \times (\partial M_e)_{C^2} \times \mathbb{C}^* \times \mathbb{C}^*\]

of \( (35) \) satisfying

- \( W' = z, \pi'((\Phi_t)(W', W'') = y, \theta_1 = \theta_2, \theta_2 \mathcal{I}_{C^*}(W) = i\alpha \).

- \( (t + i\tau)(z, z_a) \) equals the time such that \( \Gamma_y(t + i\tau) = z \).

- \( (z, W'') \in \Lambda_a^C \).

**9. L^p norms of eigenfunctions**

In §8.3 we pointed out that lower bounds on \( |||\varphi_\lambda|||_{L^1} \) lead to improved lower bounds on Hausdorff measures of nodal sets. In this section we consider general \( L^p \)-norm problems for eigenfunctions.

**9.1. Generic upper bounds on \( L^p \) norms.** We have already explained that the pointwise Weyl law \( \text{(10)} \) and remainder jump estimate \( \text{(12)} \) leads to the general sup norm bound for \( L^2 \)-normalized eigenfunctions,

\[ |||\varphi_\lambda|||_{L^\infty} \leq C_g \lambda^{\frac{m-1}{2}}, \quad (m = \dim M). \tag{136} \]

The upper bound is achieved by zonal spherical harmonics. In \[ \text{Sog} \] (see also \[ \text{Sogb, Sogb2} \]) C.D. Sogge proved general \( L^p \) bounds:

**Theorem 9.1.** (Sogge, 1985)

\[ \sup_{\varphi \in \mathcal{V}_\lambda} \| \varphi \|_p = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty \]

where

\[ \delta(p) = \begin{cases} n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2}, & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ n-1 \left( \frac{1}{2} - \frac{1}{p} \right), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \tag{138} \]
The upper bounds are sharp in the class of all \((M,g)\) and are saturated on the round sphere:

- For \(p > \frac{2(n+1)}{n-1}\), zonal (rotationally invariant) spherical harmonics saturate the \(L^p\) bounds. Such eigenfunctions also occur on surfaces of revolution.

- For \(L^p\) for \(2 \leq p \leq \frac{2(n+1)}{n-1}\) the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic. Such eigenfunctions also occur on surfaces of revolution.

The zonal has high \(L^p\) norm due to its high peaks on balls of radius \(\frac{1}{\sqrt{N}}\). The balls are so small that they do not have high \(L^p\) norms for small \(p\). The Gaussian beams are not as high but they are relatively high over an entire geodesic.

9.2. Lower bounds on \(L^1\) norms. The \(L^p\) upper bounds are the only known tool for obtaining lower bounds on \(L^1\) norms. We now prove Proposition 15:

Proof. Fix a function \(\rho \in \mathcal{S}(\mathbb{R})\) having the properties that \(\rho(0) = 1\) and \(\dot{\rho}(t) = 0\) if \(t \notin [\delta/2, \delta]\), where \(\delta > 0\) is smaller than the injectivity radius of \((M,g)\). If we then set

\[
T_\lambda f = \rho(\sqrt{-\Delta} - \lambda)f,
\]

we have that \(T_\lambda \varphi_\lambda = \varphi_\lambda\). Also, by Lemma 5.1.3 in [Sogb], \(T_\lambda\) is an oscillatory integral operator of the form

\[
T_\lambda f(x) = \lambda^{\frac{n-1}{2}} \int_M e^{i\lambda r(x,y)a_\lambda(x,y)} f(y)dy,
\]

with \(|\partial_{x,y} a_\lambda(x,y)| \leq C_\alpha\). Consequently, \(\|T_\lambda \varphi_\lambda\|_{L^\infty} \leq C\lambda^{\frac{n-1}{2}}\|\varphi_\lambda\|_{L^1}\), with \(C\) independent of \(\lambda\), and so

\[
1 = \|\varphi_\lambda\|_{L^2}^2 = \langle T \varphi_\lambda, \varphi_\lambda \rangle \leq \|T \varphi_\lambda\|_{L^\infty} \|\varphi_\lambda\|_{L^1} \leq C\lambda^{\frac{n-1}{2}}\|\varphi_\lambda\|_{L^1}.
\]

We can give another proof based on the eigenfunction estimates (Theorem 9.1), which say that

\[
\|\varphi_\lambda\|_{L^p} \leq C\lambda^{\frac{(n-1)(p-2)}{4p}}, \quad 2 < p \leq \frac{2(n+1)}{n-1}.
\]

If we pick such a \(2 < p < \frac{2(n+1)}{n-1}\), then by Hölder’s inequality, we have

\[
1 = \|\varphi_\lambda\|_{L^2}^{\frac{1}{\theta}} \leq \|\varphi_\lambda\|_{L^1} \|\varphi_\lambda\|_{L^p}^{\frac{1}{p}} \leq \|\varphi_\lambda\|_{L^1} \left( C\lambda^{\frac{(n-1)(p-2)}{4p}} \right)^{\frac{1}{\theta} - 1}, \quad \theta = \frac{p}{p-1} \left( \frac{1}{2} - \frac{1}{p} \right) = \frac{(p-2)}{2(p-1)}.
\]

which implies \(\|\varphi_\lambda\|_{L^1} \geq C\lambda^{-\frac{n-1}{4}}\), since \((1 - \frac{1}{\theta})\) of \(\frac{(n-1)(p-2)}{4p}\) is \(\frac{n-1}{4}\). \(\square\)

We remark that this lower bound for \(\|\varphi_\lambda\|_{L^1}\) is sharp on the standard sphere, since \(L^2\)-normalized highest weight spherical harmonics of degree \(k\) with eigenvalue \(\lambda^2 = k(k+n-1)\) have \(L^1\)-norms which are bounded above and below by \(k^{(n-1)/4}\) as \(k \to \infty\). Similarly, the \(L^p\)-upperbounds that we used in the second proof of this \(L^1\)-lowerbound is also sharp because of these functions.
9.3. **Riemannian manifolds with maximal eigenfunction growth.** Although the general sup norm bound \(\|\varphi\|_{L^\infty}\) is achieved by some sequences of eigenfunctions on some Riemannian manifolds (the standard sphere or a surface of revolution), it is very rare that \((M, g)\) has such sequences of eigenfunctions. We say that such \((M, g)\) have maximal eigenfunction growth. In a series of articles \([\text{SoZ}, \text{STZ}, \text{SoZ2}]\), ever more stringent conditions are given on such \((M, g)\). We now go over the results.

Denote the eigenspaces by

\[ V_\lambda = \{ \varphi : \Delta \varphi = -\lambda^2 \varphi \} \]

We measure the growth rate of \(L^p\) norms by

\[
L^p(\lambda, g) = \sup_{\varphi \in V_\lambda : \|\varphi\|_{L^2} = 1} \|\varphi\|_{L^p}.
\]

**Definition:** Say that \((M, g)\) has maximal \(L^p\) eigenfunction growth if it possesses a sequence of eigenfunctions \(\varphi_{\lambda_j}\) which saturates the \(L^p\) bounds. When \(p = \infty\) we say that it has maximal sup norm growth.

**Problem 9.2.**
- **Characterize** \((M, g)\) **with maximal** \(L^\infty\) **eigenfunction growth.** The same sequence of eigenfunctions should saturate all \(L^p\) norms with \(p \geq p_n := \frac{2(n+1)}{n-1}\).

- **Characterize** \((M, g)\) **with maximal** \(L^p\) **eigenfunction growth for** \(2 \leq p \leq \frac{2(n+1)}{n-1}\).

- **Characterize** \((M, g)\) for which \(\|\varphi_\lambda\|_{L^1} \geq C > 0\).

In \([\text{SoZ}]\), it was shown that \((M, g)\) of maximal \(L^p\) eigenfunction growth for \(p \geq p_n\) have self-focal points. The terminology is non-standard and several different terms are used.

**Definition:**
We call a point \(p\) a **self-focal point** or **blow-down point** if all geodesics leaving \(p\) loop back to \(p\) at a common time \(T\). That is, \(\exp_p T\xi = p\) (They do not have to be closed geodesics.)

We call a point \(p\) a **partial self-focal point** if there exists a positive measure in \(S^*_x M\) of directions \(\xi\) which loop back to \(p\).

The poles of a surface of revolution are self-focal and all geodesics close up smoothly (i.e. are closed geodesics). The umbilic points of an ellipsoid are self-focal but only two directions give smoothly closed geodesics (one up to time reversal).

In \([\text{SoZ}]\) is proved:
Theorem 9.3. Suppose \((M, g)\) is a \(C^\infty\) Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence \(\{\varphi_{\lambda_j}\}\) of eigenfunctions which achieves (saturates) the bound 
\[
\|\varphi_{\lambda_j}\|_{L^\infty} \geq C_0\lambda_j^{(n-1)/2}
\]
for some \(C_0 > 0\) depending only on \((M, g)\).

Then there must exist a point \(x \in M\) for which the set
\[
\mathcal{L}_x = \{ \xi \in S^*_x M : \exists T : \exp_x T\xi = x \}
\]
of directions of geodesic loops at \(x\) has positive measure in \(S^*_x M\). Here, \(\exp\) is the exponential map, and the measure \(|\Omega|\) of a set \(\Omega\) is the one induced by the metric \(g_\xi\) on \(T^*_\xi M\). For instance, the poles \(x_N, x_S\) of a surface of revolution \((S^2, g)\) satisfy \(|\mathcal{L}_x| = 2\pi\).

Theorem 9.3, Theorem 9.4, as well as the results of [SoZ, STZ], are proved by studying the remainder term \(R(\lambda, x)\) in the pointwise Weyl law,

\[
N(\lambda, x) = \sum_{j : \lambda_j \leq \lambda} |\varphi_j(x)|^2 = C_m \lambda^m + R(\lambda, x).
\]

The first term \(N(\lambda) = C_m \lambda^m\) is called the Weyl term. It is classical that the remainder is of one lower order, \(R(\lambda, x) = O(\lambda^{m-1})\). The relevance of the remainder to maximal eigenfunction growth is through the following well-known Lemma (see e.g. [SoZ]):

Lemma 9.4. Fix \(x \in M\). Then if \(\lambda \in \text{spec}\sqrt{-\Delta}\)

\[
\sup_{\varphi \in V_\lambda} \frac{|\varphi(x)|}{\|\varphi\|_2} = \sqrt{R(\lambda, x) - R(\lambda - 0, x)}.
\]

Here, for a right continuous function \(f(x)\) we denote by \(f(x + 0) - f(x - 0)\) the jump of \(f\) at \(x\). Thus, Theorem 9.3 follows from

Theorem 9.5. Let \(R(\lambda, x)\) denote the remainder for the local Weyl law at \(x\). Then

\[
R(\lambda, x) = o(\lambda^{n-1}) \text{ if } |\mathcal{L}_x| = 0.
\]

Additionally, if \(|\mathcal{L}_x| = 0\) then, given \(\varepsilon > 0\), there is a neighborhood \(\mathcal{N}\) of \(x\) and a \(\Lambda = \varepsilon < \infty\), both depending on \(\varepsilon\) so that

\[
|R(\lambda, y)| \leq \varepsilon \lambda^{n-1}, \ y \in \mathcal{N}, \ \lambda \geq \Lambda.
\]

9.4. Theorem 9.4. However, Theorem 9.3 is not sharp: on a tri-axial ellipsoid (three distinct axes), the umbilic points are self-focal points. But the eigenfunctions which maximize the sup-norm only have \(L^\infty\) norms of order \(\lambda^{n-1}/\log \lambda\). An improvement is given in [STZ].

Recently, Sogge and the author have further improved the result in the case of real analytic \((M, g)\). In this case \(|\mathcal{L}_x| > 0\) implies that \(\mathcal{L}_x = S^*_x M\) and the geometry simplifies. In [SoZ], we prove Theorem 9.4, which we restate in terms of the jumps of the remainder:

Theorem 9.6. Assume that \(U_x\) has no invariant \(L^2\) function for any \(x\). Then

\[
N(\lambda + o(1), x) - N(\lambda, x) = o(\lambda^{n-1}), \text{ uniformly in } x.
\]

Equivalently,

\[
R(\lambda + o(1), x) - R(\lambda, x) = o(\lambda^{n-1})
\]

uniformly in \(x\).
Before discussing the proof we note that the conclusion gives very stringent conditions on $(M, g)$. First, there are topological restrictions on manifolds possessing a self-focal point. If $(M, g)$ has a focal point $x_0$ then the rational cohomology $H^*(M, \mathbb{Q})$ has a single generator (Berard-Bergery). But even in this case there are many open problems:

**Problem 9.7.** All known examples of $(M, g)$ with maximal eigenfunction growth have completely integrable geodesic flow, and indeed quantum integrable Laplacians. Can one prove that maximum eigenfunction growth only occurs in the integrable case? Does it only hold if there exists a point $p$ for which $\Phi_p = Id$?

A related purely geometric problem: Do there exist $(M, g)$ with $\dim M \geq 3$ possessing self-focal points $\Phi_x \neq Id$. I.e. do there exist generalizations of umbilic points of ellipsoids in dimension two. There do not seem to exist any known examples; higher dimensional ellipsoids do not seem to have such points.

Despite these open questions, Theorem 9.7 is in a sense sharp. If there exists a self-focal point $p$ with a smooth invariant function, then one can construct a quasi-mode of order zero which lives on the flow-out Lagrangian

$$\Lambda_p := \bigcup_{t \in [0, T]} G^t S^*_p M$$

where $G^t$ is the geodesic flow and $T$ is the minimal common return time. The ‘symbol’ is the flowout of the smooth invariant density. Theorem 9.7 is valid for quasi-modes as well as eigenfunctions. Indeed, most microlocal methods cannot distinguish modes and quasi-modes.

**Proposition 9.8.** Suppose that $(M, g)$ has a point $p$ which is a self-focal point whose first return map $\Phi_x$ at the return time $T$ is the identity map of $S^*_p M$. Then there exists a quasi-model of order zero associated to the sequence $\{2T^k + \frac{\theta}{2} : k = 1, 2, 3, \ldots \}$ which concentrates microlocally on the flow-out of $S^*_p M$. (See 9.8 for background and more precise information).

**9.5. Sketch of proof of Theorem TL14.** We first outline the proof. A key issue is the uniformity of remainder estimates of $R(\lambda, x)$ as $x$ varies. Intuitively it is obvious that the main points of interest are the self-focal points. But at this time of writing, we cannot exclude the possibility, even in the real analytic setting, that there are an infinite number of such points with twisted return maps. Points which isolated from the set of self-focal points are easy to deal with, but there may be non-self-focal points which lie in the closure of the self-focal points. We introduce some notation.

**Definition:** We say that $x \in M$

- is an $\mathcal{L}$ point $(x \in \mathcal{L})$ if $\mathcal{L}_x = \pi^{-1}(x) \simeq S^*_x M$. Thus, $x$ is a self-focal point.
- is a $\mathcal{CL}$ point $(x \in \mathcal{CL})$ if $x \in \mathcal{L}$ and $\Phi_x = Id$. Thus, all of the loops at $x$ are smoothly closed.
- is a $\mathcal{T CL}$ point $(x \in \mathcal{T CL})$ if $x \in \mathcal{L}$ but $\Phi_x \neq Id$, i.e. $x$ is a twisted self-focal point. Equivalently, $\mu_x \{\xi \in \mathcal{L}_x : \Phi_x(\xi) = \xi\} = 0$; All directions are loop directions, but almost none are directions of smoothly closed loops.

To prove Theorem TL14 we may (and henceforth will) assume that $\mathcal{CL} = \emptyset$. Thus, $\mathcal{L} = \mathcal{T CL}$. We also let $\overline{\mathcal{L}}$ denote the closure of the set of self-focal points. At this time of writing, we do not know how to exclude that $\overline{\mathcal{L}} = M$, i.e. that the set of self-focal points is dense.
**Problem 9.9.** Prove (or disprove) that if $\mathcal{L} = \emptyset$ and if $(M, g)$ is real analytic, then $\mathcal{L}$ is a finite set.

We also need to further specify times of returns. It is well-known and easy to prove that if all $\xi \in \pi^{-1}(x)$ are loop directions, then the time $T(x, \xi)$ of first return is constant on $\pi^{-1}(x)$. This is because an analytic function is constant on its critical point set.

**Definition:** We say that $x \in M$

- is a $T\mathcal{L}_T$ point $(x \in T\mathcal{L}_T)$ if $x \in T\mathcal{L}$ and if $T(x, \xi) \leq T$ for all $\xi \in \pi^{-1}(x)$. We denote the set of such points by $T\mathcal{L}_T$.

**Lemma 9.10.** If $(M, g)$ is real analytic, then $T\mathcal{L}_T$ is a finite set.

There are several ways to prove this. One is to consider the set of all loop points, $\mathcal{E} = \{(x, \xi) \in TM : \exp_x \xi = x\}$, where as usual we identify vectors and co-vectors with the metric. Then at a self-focal point $p$, $\mathcal{E} \cap T_pM$ contains a union of spheres of radii $kT(p)$, $k = 1, 2, 3, \ldots$. The condition that $\Phi_p \neq I$ can be used to show that each sphere is a component of $\mathcal{E}$, i.e. is isolated from the rest of $\mathcal{E}$. Hence in the compact set $B^*_T M = \{(x, \xi) : |\xi| \leq T\}$, there can only exist a finite number of such components. Another way to prove it is to show that any limit point $x$, with $p_j \to x$ and $p_j \in T\mathcal{L}_T$ must be a $T\mathcal{L}_T$ point whose first return map is the identity. Both proofs involve the study of Jacobi fields along the looping geodesics.

To outline the proof, let $\hat{\rho} \in C_0^\infty$ be an even function with $\hat{\rho}(0) = 1$, $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, and $\hat{\rho}_T(t) = \hat{\rho}(\frac{t}{T})$. The classical cosine Tauberian method to determine Weyl asymptotics is to study

One starts from the smoothed spectral expansion $\mathbb{R}$

$$
\rho_T * dN(\lambda, x) = \int_{\mathbb{R}} \hat{\rho}(\frac{t}{T}) e^{i\lambda t} U(t, x, x) dt
$$

$$
= a_0 \lambda^{n-1} + a_1 \lambda^{n-2} + \lambda^{n-1} \sum_{j=1}^{\infty} R_j(\lambda, x, T) + o_T(\lambda^{n-1}),
$$

with uniform remainder in $x$. The sum over $j$ is a sum over charts needed to parametrize the canonical relation of $U(t, x, y)$, i.e. the graph of the geodesic flow. By the usual parametrix construction for $U(t) = e^{it\sqrt{\Delta}}$, one proves that there exist phases $\tilde{t}_j$ and amplitudes $a_{j0}$ such that

$$
R_j(\lambda, x, T) \simeq \lambda^{n-1} \int_{S^*_x M} e^{i\lambda \tilde{t}_j(x, \xi)} ((\hat{\rho} a_{j0}) (|d\xi|) + O(\lambda^{n-2}).
$$

As in $\mathbb{R}$

$$
\text{exercise 7. Show that } \rho_T * dN(\lambda, x) \text{ is a semi-classical Lagrangian distribution in the sense of §13. What is its principal symbol?}
$$

To illustrate the notation, we consider a flat torus $\mathbb{R}^n / \Gamma$ with $\Gamma \subset \mathbb{R}^n$ a full rank lattice. As is well-known, the wave kernel then has the form

$$
U(t, x, y) = \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^n} e^{i(x-y-\gamma \xi)} e^{it|\xi|} d\xi.
$$
Thus, the indices \( j \) may be taken to be the lattice points \( \gamma \in \Gamma \), and

\[
\rho_T \ast dN(\lambda, x) = \sum_{\gamma \in \Gamma} \int_{\mathbb{R}} \int_0^\infty \int_{S_n-1} \hat{\rho}(\frac{\xi}{T}) e^{i\xi(\gamma, \omega)} e^{itr} e^{-it\lambda r} \cdot d\rho \cdot dr \cdot dt \cdot d\omega
\]

We change variables \( r \to \lambda r \) to get a full phase \( \lambda(r\gamma, \omega) + tr - t \). The stationary phase points in \((r, t)\) are \((\gamma, \omega) = t\) and \(r = 1\). Thus,

\[
\tilde{t}_\gamma(x, \omega) = \langle \gamma, \omega \rangle.
\]

The geometric interpretation of \( t^*_\gamma(x, \omega) \) is that it is the value of \( t \) for which the geodesic \( exp_x t\omega = x + t\omega \) comes closest to the representative \( x + \gamma \) of \( x \) in the \( \gamma \)th chart. Indeed, the line \( x + t\omega \) is ‘closest’ to \( x + \gamma \) when \( t\omega \) closest to \( \gamma \), since \(|\gamma - t\omega|^2 = |\gamma|^2 - 2t\langle \gamma, \omega \rangle + t^2\). On a general \((M, g)\) without conjugate points,

\[
\tilde{t}_\gamma(x, \omega) = \langle \exp_x^{-1} \gamma x, \omega \rangle.
\]

9.6. Size of the remainder at a self-focal point. The first key observation is that \( P_1 \) takes a special form at a self-focal point. At a self-focal point \( x \) define \( U_x \) as in \((\ref{ux})\). Also define

(148) \[
U_x^\pm(\lambda) = e^{i\lambda t^\pm} U_x^\pm.
\]

The following observation is due to Safarov \([\text{Saf}]\) (see also \([\text{SV}]\)).

**Lemma 9.11.** Suppose that \( x \) is a self-focal point. If \( \hat{\rho} = 0 \) in a neighborhood of \( t = 0 \) then

(149) \[
\rho_T' \ast N(\lambda, x) = \lambda^{n-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{S^*_x M} \hat{\rho}(\frac{kT(\xi)}{T}) U_x(\lambda)^k \cdot \frac{1}{d\xi} + O(\lambda^{n-2}).
\]

Here is the main result showing that \( R(\lambda, x) \) is small at the self-focal points if there do not exist invariant \( L^2 \) functions. \( T^{(k)}_x(\xi) \) is the \( k \)th return time of \( \xi \) for \( \Phi_x \).

**Proposition 9.12.** Assume that \( x \) is a self-focal point and that \( U_x \) has no invariant \( L^2 \) function. Then, for all \( \eta > 0 \), there exists \( T \) so that

(150) \[
\frac{1}{T} \left| \int_{S^*_x M} \sum_{k=0}^\infty \hat{\rho}(\frac{T^{(k)}_x(\xi)}{T}) U_x^k \cdot \frac{1}{d\xi} \right| \leq \eta.
\]

This is a simple application of the von Neumann mean ergodic theorem to the unitary operator \( U_x \). Indeed, \( \frac{1}{N} \sum_{k=0}^N U_x^k \to P_x \), where \( P_x : L^2(S^*_x M) \to L^2_0(S^*_x M) \) is the orthogonal projections onto the invariant \( L^2 \) functions for \( U_x \). By our assumption, \( P_x = 0 \).

Proposition 9.12 is not apriori uniform as \( x \) varies over self-focal points, since there is no obvious relation between \( \Phi_x \) at one self-focal point and another. It would of course be uniform if we knew that there only exist a finite number of self-focal points. As mentioned above, this is currently unknown. However, there is a second mechanism behind Proposition 9.12. Namely, if the first common return time \( T^{(1)}_x(\xi) \) is larger than \( T \), then there is only one term \( k = 0 \) in the sum with the cutoff \( \hat{\rho}_T \) and the sum is \( O(\frac{1}{T}) \).
9.7. Decomposition of the remainder into almost loop directions and far from loop directions. We now consider non-self-focal points. Then the function \( \tilde{t}_j(x, \xi) \) has almost no critical points in \( S^*_x M \).

Pick \( f \geq 0 \in C^\infty(\mathbb{R}) \) which equals 1 on \( |s| \leq 1 \) and zero for \( |s| \geq 2 \) and split up the \( j \)th term into two terms using \( f(\epsilon^{-2}|\nabla_\xi \tilde{t}_j|^2) \) and \( 1 - f(\epsilon^{-2}|\nabla_\xi \tilde{t}_j|^2) \):

\[
R_j(\lambda, x, T) = R_{j1}(\lambda, x, T, \epsilon) + R_{j2}(\lambda, x, T, \epsilon),
\]

where

\[
R_{j1}(\lambda, x, T, \epsilon) := \int_{S^*_x M} e^{i\lambda \tilde{t}_j} f(\epsilon^{-1}|\nabla_\xi \tilde{t}_j(x, \xi)|^2)(\hat{\rho}(T_x(\xi))) a_0(T_x(\xi), x, \xi) d\xi
\]

The second term \( R_{j2} \) comes from the \( 1 - f(\epsilon^{-2}|\nabla_\xi T_x(\xi)|^2) \) term. By one integration by parts, one easily has

**Lemma 9.13.** For all \( T > 0 \) and \( \epsilon \geq \lambda^{-\frac{1}{2}} \log \lambda \) we have:

\[
\sup_{x \in M} |R_2(\lambda, x, T, \epsilon)| \leq C(\epsilon^2 \lambda)^{-1}.
\]

The \( f \) term involves the contribution of the almost-critical points of \( \tilde{t}_j \). They are estimated by the measure of the almost-critical set.

**Lemma 9.14.** There exists a uniform positive constant \( C \) so that for all \( (x, \epsilon) \),

\[
|R_{j1}(x, \epsilon)| \leq C \mu_x \left( \{ \xi : 0 < |\nabla_\xi \tilde{t}_j(x, \xi)|^2 < \epsilon^2 \} \right),
\]

9.8. Points in \( M \setminus \overline{T_L} \). If \( x \) is isolated from \( \overline{T_L} \) then there is a uniform bound on the size of the remainder near \( x \).

**Lemma 9.15.** Suppose that \( x \notin \overline{T_L} \). Then given \( \eta > 0 \) there exists a ball \( B(x, r(x, \eta)) \) with radius \( r(x, \eta) > 0 \) and \( \epsilon > 0 \) so that

\[
\sup_{y \in B(x, r(x, \eta))} |R(\lambda, y, \epsilon)| \leq \eta.
\]

Indeed, we pick \( r(x, \eta) \) so that the closure of \( B(x, r(x, \eta)) \) is disjoint from \( \overline{T_L} \). Then the one-parameter family of functions \( F_\epsilon(y) = \mu_y \left( \{ \xi \in S^*_y M : 0 < |\nabla_\xi \tilde{t}_j(y, \xi)|^2 < \epsilon^2 \} \right) \) is decreasing to zero as \( \epsilon \to 0 \) for each \( y \). By Dini’s theorem, the family tends to zero uniformly on \( B(x, r(x, \eta)) \).

9.9. Perturbation theory of the remainder. So far, we have good remainder estimates at each self-focal point and in balls around points isolated from the self-focal points. We still need to deal with the uniformity issues as \( p \) varies among self-focal points and points in \( \overline{T_L} \). We now compare remainders at nearby points. Although \( R_j(\lambda, x, T) \) is oscillatory, the estimates on \( R_{j1} \) and the ergodic estimates do not use the oscillatory factor \( e^{i\lambda t} \), which in fact is only used in Lemma 6.13. Hence we compare absolute remainders \( |R|(x, T) \), i.e. where we take the absolute under the integral sign. They are independent of \( \lambda \). The integrands of the remainders vary smoothly with the base point and only involve integrations over different fibers \( S^*_x M \) of \( S^* M \to M \).

**Lemma 9.16.** We have,

\[
||R|(x, T) - |R|(y, T)| \leq C e^{aT} \text{dist}(x, y).
\]
Indeed, we write the difference as the integral of its derivative. The derivative involves the change in \( \Phi^n_x \) as \( x \) varies over iterates up to time \( T \) and therefore is estimated by the sup norm \( e^{aT} \) of the first derivative of the geodesic flow up to time \( T \). If we choose a ball of radius \( \delta e^{-aT} \) around a focal point, we obtain'

**Corollary 9.17.** For any \( \eta > 0, T > 0 \) and any focal point \( p \in T\mathcal{L} \) there exists \( r(p, \eta) \) so that

\[
\sup_{y \in B(p, r(p, \eta))} |R(\lambda, y, T)| \leq \eta.
\]

To complete the proof of Theorem 9.14 we prove

**Lemma 9.18.** Let \( x \in T\mathcal{L} \setminus T\mathcal{L} \). Then for any \( \eta > 0 \) there exists \( r(x, \eta) > 0 \) so that

\[
\sup_{y \in B(x, r(x, \eta))} |R(\lambda, y, T)| \leq \eta.
\]

Indeed, let \( p_j \to x \) with \( T(p_j) \to \infty \). Then the remainder is given at each \( p_j \) by the left side of (149). But for any fixed \( T \), the first term of (149) has at most one term for \( j \) sufficiently large. Since the remainder is continuous, the remainder at \( x \) is the limit of the remainders at \( p_j \) and is therefore \( O(T^{-1}) + O(\lambda^{-1}) \).

By the perturbation estimate, one has the same remainder estimate in a sufficiently small ball around \( x \).

**9.10. Conclusions.**

- No eigenfunction \( \varphi_j(x) \) can be maximally large at a point \( x \) which is \( \geq \lambda_j^{-\frac{1}{2}} \log \lambda_j \) away from the self-focal points.

- When there are no invariant measures, \( \varphi_j \) also cannot be large at a self-focal point.

- If \( \varphi_j \) is not large at any self-focal point, it is also not large near a self-focal point.

**10. Appendix: Proof of Theorem 9.14 To appear in my CBMS lectures**

In this section, we sketch a proof of the upper bound of Theorem 9.14 using analytic continuations of eigenfunctions and pluri-subharmonic theory on Grauert tubes. As in [DF] the proof is based on Crofton’s formula, integral geometry and the growth rates of complexified eigenfunctions. The details are somewhat different.

We start with the integral geometric approach of [DF] (Lemma 6.3) (see also [L] (3.21) and [H], Proof of Theorem 2.2.1). There exists a “Crofton formula” in the real domain which bounds the local nodal hypersurface volume above,

\[
\mathcal{H}^{n-1}(\mathcal{N}_{\varphi, \lambda} \cap U) \leq C_L \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi, \lambda}) d\mu(\ell)
\]

by a constant \( C_L \) times the average over all line segments of length \( L \) in a local coordinate patch \( U \) of the number of intersection points of the line with the nodal hypersurface.

The complexification of a real line \( \ell = x + \mathbb{R}v \) with \( x, v \in \mathbb{R}^n \) is \( \ell_{\mathbb{C}} = x + \mathbb{C}v \). Since the number of intersection points (or zeros) only increases if we count complex intersections, we have

\[
\int_{\mathcal{L}} \#(\mathcal{N}_{\varphi, \lambda} \cap \ell) d\mu(\ell) \leq \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi, \lambda}^{\mathbb{C}} \cap \ell_{\mathbb{C}}) d\mu(\ell).
\]
Hence to prove Theorem \(DF\), it suffices to show

\[ \mathcal{H}^{n-1}(\mathcal{N}_\lambda) \leq C_L \int_{\mathcal{C}} \#(\mathcal{N}_\lambda)^C \cap \ell(\ell) d\mu(\ell) \leq C\lambda. \]

10.1. Hausdorff measure and Crofton formula for real geodesic arcs. A Crofton formula arises from a double fibration

\[ I \]

\[ \pi_1 \quad \Gamma \quad \pi_2 \]

\[ \Gamma \quad B, \]

where \( \Gamma \) parametrizes a family of submanifolds \( B_\gamma \) of \( B \). The points of \( b \) then parametrize a family of submanifolds \( \Gamma_b = \{ \gamma \in \Gamma : b \in B_\gamma \} \) and the top space is the incidence relation in \( B \times \Gamma \) that \( b \in \gamma \).

We would like to define \( \Gamma \) as the space of geodesics of \((M,g)\), i.e. the space of orbits of the geodesic flow on \( S^*M \). Heuristically, the space of geodesics is the quotient space \( S^*M/R \) where \( R \) acts by the geodesic flow \( G_t \) (i.e. the Hamiltonian flow of \( H \)). Of course, for a general (i.e. non-Zoll) \((M,g)\) the ‘space of geodesics’ is not a Hausdorff space and so we do not have a simple analogue of the space of lines in \( \mathbb{R}^n \). Instead we consider the \( G_T \) of geodesic arcs of length \( T \). If we only use partial orbits of length \( T \), no two partial orbits are equivalent and the space of geodesic arcs \( \gamma_{T,x}^{T,\xi} \) of length \( T \) is simply parametrized by \( S^*M \).

Hence we let \( B = S^*M \) and also \( G_T \simeq S^*M \). The fact that different arcs of length \( T \) of the same geodesic are distinguished leads to some redundancy.

As before, we denote by \( d\mu_L \) the Liouville measure on \( S^*M \). We also denote by \( \omega \) the standard symplectic form on \( T^*M \) and by \( \alpha \) the canonical one form. Then \( d\mu_L = \omega^{n-1} \wedge \alpha \) on \( S^*M \). Indeed, \( d\mu_L \) is characterized by the formula \( d\mu_L \wedge dH = \omega^n \), where \( H(x,\xi) = |\xi|_g \).

So it suffices to verify that \( \alpha \wedge dH = \omega \) on \( S^*M \). We take the interior product \( \iota_{\xi_H} \) with the Hamilton vector field \( \xi_H \) on both sides, and the identity follows from the fact that \( \alpha(\xi_H) = \sum \xi_j \frac{\partial H}{\partial \xi_j} = H \) on \( S^*M \), since \( H \) is homogeneous of degree one.

In the following, let \( L_1 \) denote the length of the shortest closed geodesic of \((M,g)\).

**Lemma 10.2.** Let \( N \subset M \) be any smooth hypersurface. Then for \( T \geq L_1 \),

\[ \mathcal{H}^{n-1}(N) = \frac{\beta_n}{T} \int_{S^*M} \# \{ t \in [-T,T] : G^t(x,\omega) \in S^*_N M \} d\mu_L(x,\omega), \]

where \( \beta_n \) is \( 2(n-1)! \) times the volume of the unit ball in \( \mathbb{R}^{n-2} \).

**10.2. Proof of Theorem \(DF\).**

**Proof.** We complexify the Lagrange immersion \( \iota \) from a line (segment) to a strip in \( \mathbb{C} \): Define

\[ \psi : S_\epsilon \times S^*M \to M_\mathbb{C}, \quad \psi(t + i\tau, x, v) = \exp_x(t + i\tau)v, \quad |\tau| \leq \epsilon \]

By definition of the Grauert tube, \( \psi \) is surjective onto \( M_\epsilon \). For each \((x, v) \in S^*M\),

\[ \psi_{x,v}(t + i\tau) = \exp_x(t + i\tau)v \]
is a holomorphic strip. Here, \( S_{\epsilon} = \{ t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon \} \). We also denote by \( S_{\epsilon,L} = \{ t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon, |t| \leq L \} \).

Since \( \psi_{x,v} \) is a holomorphic strip,
\[
\psi_{x,v}(1/\chi dd^c \log |\phi_j^c|^2) = 1/\chi dd^c_{t+i\tau} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) v = 1/\chi \sum_{t+i\tau; \phi_j^c(\exp_x(t+i\tau)v) = 0} \delta_{t+i\tau}.
\]

Put:
\[
A_{L,\epsilon}(1/\chi dd^c \log |\phi_j^c|^2) = 1/\chi \int_{S^{*} M} \int_{S_{\epsilon,L}} dd^c_{t+i\tau} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) \nu_L(x,v).
\]

Then by Lemma \[10.2\],
\[
A_{L,\epsilon}(1/\chi dd^c \log |\phi_j^c|^2) = 1/\chi \int_{S^{*} M} \#(N^c_{\phi_L} \cap \psi_{x,v}(S_{\epsilon,L})) \mu(x,v) \]
\[
\geq 1/\chi H^{n-1}(N_{\phi_L}).
\]

Since \( dd^c_{t+i\tau} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) \) is a positive \((1,1)\) form on the strip, the integral over \( S_{\epsilon} \) is only increased if we integrate against a positive smooth test function \( \chi \in C^\infty_{c}(\mathbb{C}) \) which equals one on \( S_{\epsilon,L} \) and vanishes off \( S_{2\epsilon,L} \). Integrating by parts the \( dd^c \) onto \( \chi \), we have
\[
A_{L,\epsilon}(1/\chi dd^c \log |\phi_j^c|^2) \leq 1/\chi \int_{S^{*} M} \int_{C} dd^c_{t+i\tau} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) \chi(t + i\tau) \nu_L(x,v) \]
\[
= 1/\chi \int_{S^{*} M} \int_{C} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) dd^c_{t+i\tau} \chi(t + i\tau) \nu_L(x,v).
\]

Now write \( \log |x| = \log_+ |x| - \log_- |x| \). Here \( \log_+ |x| = \max\{0, \log |x|\} \) and \( \log_- |x| = \max\{0, -\log |x|\} \). Then we need upper bounds for
\[
1/\chi \int_{S^{*} M} \int_{C} \log_+ |\phi_j^c|^2(\exp_x(t + i\tau)v) dd^c_{t+i\tau} \chi(t + i\tau) \nu_L(x,v).
\]

For \( \log_+ \) the bound is an immediate consequence of Proposition \[5.2\]. For \( \log_- \) the bound is subtler: we need to show that \( |\phi_j^c(z)| \) cannot be too small on too large a set. We recall that \( \exp_x(t + i\tau)v : S^{*} M \times S_{\epsilon,L} \to M_\tau \) is for any fixed \( t, \tau \) a diffeomorphism and so the map
\[
E : (t + i\tau, x, v) \in \times S_{\epsilon,L} \to M_\tau \to M_\tau
\]
is a smooth fibration with strip fibers. Pushing forward the measure \( dd^c_{t+\tau} \chi(t+i\tau) \nu_L(x,v) \) gives us a positive measure \( d\mu \) on \( M_\tau \). In fact, \( d\mu_L \) is equivalent under \( E \) to the contact volume form \( \alpha \wedge \omega^{n-1}_p \) where \( \alpha = d^c \sqrt{\rho} \). Over a point \( \zeta \in M_\tau \) the fiber of the map is a geodesic arc
\[
\{(t + i\tau, x, v) : \exp_x(t + i\tau)v = \zeta, \ \tau = \sqrt{\rho}(\zeta)\}.
\]

Since the Kähler volume form is \( d\tau \) times the contact volume form, the pushfoward equals the Kähler volume form times the integral of \( \Delta_{t+i\tau}\chi \) over the arc. Thus it is a smooth (and of course signed) multiple \( J \) of the Kähler volume form \( dV \). We then have
\[
\int_{S^{*} M} \int_{C} \log |\phi_j^c|^2(\exp_x(t + i\tau)v) dd^c_{t+i\tau} \chi(t + i\tau) \nu_L(x,v) = \int_{M_\tau} \log |\phi_j^c|^2 J dV.
\]
We thus to prove that the right side is \( \geq -C\lambda \) for some \( C > 0 \).
This Lemma implies the desired lower bound on \( \| \varphi \|_{L^p(M)} \): there exists \( C > 0 \) so that

\[
\frac{1}{\lambda} \int_{M_r} \log |\varphi| JdV \geq -C.
\]

For if not, there exists a subsequence of eigenvalues \( \lambda_{j_k} \) so that \( \frac{1}{\lambda_{j_k}} \int_{M_r} \log |\varphi_{\lambda_{j_k}}| JdV \to -\infty \).

But the family \( \{ \frac{1}{\lambda} \log |\varphi_{\lambda_{j_k}}| \} \) certainly has a uniform upper bound. Moreover the sequence does not tend uniformly to \( -\infty \) since \( ||\varphi||_{L^2(M)} = 1 \). It follows that a further subsequences tends in \( L^1 \) to a limit \( u \) and by the dominated convergence theorem the limit of \( (157) \) along the sequence equals \( \int_{M_r} u JdV \neq -\infty \). This contradiction concludes the proof of \( (157) \) and hence the theorem.

\[
\square
\]

11. Appendix on Spherical Harmonics

Spherical harmonics furnish the extremals for \( L^p \) norms of eigenfunctions \( \varphi_\lambda \) as \((M,g)\) ranges over Riemannian manifolds and \( \varphi_\lambda \) ranges over its eigenfunctions. They are not unique in this respect: surfaces of revolution and their higher dimensional analogues also give examples where extremal eigenfunction bounds are achieved. In this appendix we review the definition and properties of spherical harmonics.

Eigenfunctions of the Laplacian \( \Delta_{S^n} \) on the standard sphere \( S^n \) are restrictions of harmonic homogeneous polynomials on \( \mathbb{R}^{n+1} \).

Let \( \Delta_{\mathbb{R}^{n+1}} = -\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2} \right) \) denote the Euclidean Laplacian. In polar coordinates \((r,\omega)\) on \( \mathbb{R}^{n+1} \), we have \( \Delta_{\mathbb{R}^{n+1}} = -\left( \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^n} \). A polynomial \( P(x) = P(x_1,\ldots,x_{n+1}) \) on \( \mathbb{R}^{n+1} \) is called:

- homogeneous of degree \( k \) if \( P(rx) = r^k P(x) \). We denote the space of such polynomials by \( \mathcal{P}_k \). A basis is given by the monomials
  \[ x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_{n+1} = k. \]
- Harmonic if \( \Delta_{\mathbb{R}^{n+1}} P(x) = 0 \). We denote the space of harmonic homogeneous polynomials of degree \( k \) by \( \mathcal{H}_k \).

Suppose that \( P(x) \) is a homogeneous harmonic polynomial of degree \( k \) on \( \mathbb{R}^{n+1} \). Then,

\[
0 = \Delta_{\mathbb{R}^{n+1}} P = - \left\{ \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} \right\} r^k P(\omega) + \frac{1}{r^2} \Delta_{S^n} P(\omega)
\]

\[
\implies \Delta_{S^n} P(\omega) = (k(k-1) + nk) P(\omega).
\]

Thus, if we restrict \( P(x) \) to the unit sphere \( S^n \) we obtain an eigenfunction of eigenvalue \( k(n+k-1) \). Let \( \mathcal{H}_k \subset L^2(S^n) \) denote the space of spherical harmonics of degree \( k \). Then:

- \( L^2(S^n) = \bigoplus_{k=0}^\infty \mathcal{H}_k \). The sum is orthogonal.
- \( Sp(\Delta_{S^n}) = \{\lambda_k^2 = k(n+k-1)\} \).
- \( \dim\mathcal{H}_k \) is given by
  \[
d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}.
\]
The Laplacian $\Delta_{S^n}$ is quantum integrable. For simplicity, we restrict to $S^2$. Then the group $SO(2) \subset SO(3)$ of rotations around the $x_3$-axis commutes with the Laplacian. We denote its infinitesimal generator by $L_3 = \frac{\partial}{\partial \theta}$. The standard basis of spherical harmonics is given by the joint eigenfunctions ($|m| \leq k$)

\[
\begin{cases}
\Delta_{S^2} Y^k_m = k(k + 1) Y^k_m; \\
\frac{\partial}{\partial \theta} Y^k_m = m Y^k_m.
\end{cases}
\]

Two basic spherical harmonics are:

- The highest weight spherical harmonic $Y^k_k$. As a homogeneous polynomial it is given up to a normalizing constant by $(x_1 + ix_2)^k$ in $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$. It is a 'Gaussian beam' along the equator $\{x_3 = 0\}$, and is also a quasi-mode associated to this stable elliptic orbit.

- The zonal spherical harmonic $Y^k_0$. It may be expressed in terms of the orthogonal projection $\Pi_k : L^2(S^2) \to \mathcal{H}_k$. 

We now explain the last statement: For any \( n \), the kernel \( \Pi_k(x, y) \) of \( \Pi_k \) is defined by

\[
\Pi_k f(x) = \int_{S^n} \Pi_k(x, y) f(y) dS(y),
\]

where \( dS \) is the standard surface measure. If \( \{ Y^k_m \} \) is an orthonormal basis of \( \mathcal{H}_k \) then

\[
\Pi_k(x, y) = \sum_{m=1}^{d_k} Y^k_m(x) Y^k_m(y).
\]

Thus for each \( y \), \( \Pi_k(x, y) \in \mathcal{H}_k \). We can \( L^2 \) normalize this function by dividing by the square root of

\[
||\Pi_k(\cdot, y)||_{L^2}^2 = \int_{S^n} \Pi_k(x, y) \Pi_k(y, x) dS(x) = \Pi_k(y, y).
\]

We note that \( \Pi_k(y, y) = C_k \) since it is rotationally invariant and \( O(n+1) \) acts transitively on \( S^n \). Its integral is \( \dim \mathcal{H}_k \), hence, \( \Pi_k(y, y) = \frac{1}{\text{Vol}(S^n)} \dim \mathcal{H}_k \). Hence the normalized projection kernel with ‘peak’ at \( y_0 \) is

\[
Y^k_0(x) = \frac{\Pi_k(x, y_0) \sqrt{\text{Vol}(S^n)}}{\sqrt{\dim \mathcal{H}_k}}.
\]

Here, we put \( y_0 \) equal to the north pole \((0, 0 \cdots, 1)\). The resulting function is called a zonal spherical harmonic since it is invariant under the group \( O(n + 1) \) of rotations fixing \( y_0 \).

One can rotate \( Y^k_0(x) \) to \( Y^k_0(g \cdot x) \) with \( g \in O(n + 1) \) to place the ‘pole’ or ‘peak point’ at any point in \( S^2 \).

12. Appendix: Wave equation and Hadamard parametrix

The Cauchy problem for the wave equation on \( \mathbb{R} \times M \) is the initial value problem (with Cauchy data \( f, g \) )

\[
\begin{cases}
\Box u(t, x) = 0, \\
u(0, x) = f, \quad \frac{\partial}{\partial t} u(0, x) = g(x),
\end{cases}
\]
The solution operator of the Cauchy problem (the “propagator”) is the wave group,

\[ U(t) = \begin{pmatrix} \cos t \sqrt{\Delta} & \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}} \\ \sqrt{\Delta} \sin t \sqrt{\Delta} & \cos t \sqrt{\Delta} \end{pmatrix} \]

The solution of the Cauchy problem with data \((f, g)\) is \(U(t) \begin{pmatrix} f \\ g \end{pmatrix}\).

- Even part \(\cos t \sqrt{\Delta}\) which solves the initial value problem

\[
\begin{cases}
(\frac{\partial^2}{\partial t^2} - \Delta) u = 0 \\
u|_{t=0} = f \\
\frac{\partial}{\partial t} u|_{t=0} = 0
\end{cases}
\]

- Odd part \(\frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}}\) is the operator solving

\[
\begin{cases}
(\frac{\partial^2}{\partial t^2} - \Delta) u = 0 \\
u|_{t=0} = 0 \\
\frac{\partial}{\partial t} u|_{t=0} = g
\end{cases}
\]

The forward half-wave group is the solution operator of the Cauchy problem

\[
(\frac{1}{i} \frac{\partial}{\partial t} - \sqrt{-\Delta}) u = 0, \quad u(0, x) = u_0.
\]

The solution is given by

\[ u(t, x) = U(t) u_0(x), \]

with

\[ U(t) = e^{it \sqrt{-\Delta}} \]

the unitary group on \(L^2(M)\) generated by the self-adjoint elliptic operator \(\sqrt{-\Delta}\).

A fundamental solution of the wave equation is a solution of

\[ \Box E(t, x, y) = \delta_0(t) \delta_x(y). \]

The right side is the Schwartz kernel of the identity operator on \(\mathbb{R} \times M\).

There exists a unique fundamental solution with support in the forward light cone, called the advanced (or forward) propagator. It is given by

\[ E_+(t) = H(t) \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}}, \]

where \(H(t) = 1_{t \geq 0}\) is the Heaviside step function.

12.1. Hormander parametrix. We would like to construct a parametrix of the form

\[
\int_{T^* M} e^{i(\exp^{-1}_y x, \eta)} e^{it|\eta|} A(t, x, y, \eta) d\eta.
\]

This is a homogeneous Fourier integral operator kernel (see §1.3).

Hörmander actually constructs one of the form

\[
\int_{T^* M} e^{i\psi(x, y, \eta)} e^{it|\eta|} A(t, x, y, \eta) d\eta,
\]

where \(\psi(x, y, \eta)\) is a suitable phase function.
where \( \psi \) solves the Hamilton Jacobi Cauchy problem,
\[
\begin{aligned}
q(x, d_x \psi(x, y, \eta)) &= q(y, \eta), \\
\psi(x, y, \eta) &= 0 \iff \langle x - y, \eta \rangle = 0, \\
d_x \psi(x, y, \eta) &= \eta, \quad (\text{for } x = y).
\end{aligned}
\]

The question is whether \( \langle \exp_{y}^{-1} x, \eta \rangle \) solves the equations for \( \psi \). Only the first one is unclear. We need to understand \( \nabla_x \langle \exp_{y}^{-1} x, \eta \rangle \). We are only interested in the norm of the gradient at \( x \) but it is useful to consider the entire expression. If we write \( \eta = \rho \omega \) with \( |\omega|_y = 1 \), then \( \rho \) can be eliminated from the equation by homogeneity. We fix \( (y, \eta) \in S_\theta^g M \) and consider \( \exp_y : T_y M \to M. \) We wish to vary \( \exp_{y}^{-1} x(t) \) along a curve. Now the level sets of \( \langle \exp_{y}^{-1} x, \eta \rangle \) define a notion of local ‘plane waves’ of \( (M, g) \) near \( y \). They are actual hyperplanes normal to \( \omega \) in flat \( \mathbb{R}^n \) and in any case are far different from distance spheres. Having fixed \( (y, \eta) \), \( \nabla_x \langle \exp_{y}^{-1} x, \omega \rangle \) are normal to the plane waves defined by \( (y, \eta) \). To determine the length we need to see how \( \nabla_x \langle \exp_{y}^{-1} x, \omega \rangle \) changes in directions normal to plane waves.

The level sets of \( \langle \exp_{y}^{-1} x, \eta \rangle \) are images under \( \exp_y \) of level sets of \( \langle \xi, \eta \rangle = C \) in \( T_y M. \) These are parallel hyperplanes normal to \( \eta \). The radial geodesic in the direction \( \eta \) is of course normal to the exponential image of the hyperplanes. Hence, this radial geodesic is parallel to \( \langle \exp_{y}^{-1} x, \eta \rangle \) when \( \exp_y t \eta = x. \) It follows that \( |\nabla_x \langle \exp_{y}^{-1} x, \eta \rangle| \) at this point equals \( \frac{\partial}{\partial t} \langle \exp_{y}^{-1} \exp_y t \eta, \eta \rangle = t |\eta|_y. \) Hence \( |\nabla_x \langle \exp_{y}^{-1} x, \eta \rangle|_x = 1 \) at such points.

### 12.2. Wave group: \( r^2 - t^2 \)

The wave group of a Riemannian manifold is the unitary group \( U(t) = e^{it\sqrt{\Delta}} \) where \( \Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} g^{ij} \frac{\partial}{\partial x_j} \) is the Laplacian of \( (M, g) \). Here, \( g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \), \([g^{ij}]\) is the inverse matrix to \([g_{ij}]\) and \( g = \det[g_{ij}] \). On a compact manifold, \( \Delta \) has a discrete spectrum
\[
\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}
\]
of eigenvalues and eigenfunctions. The (Schwartz) kernel of the wave group can be constructed in two very different ways: in terms of the spectral data
\[
U(t)(x, y) = \sum_j e^{it\lambda_j} \varphi_j(x) \varphi_j(y).
\]

We now review the construction of a Hadamard parametrix,
\[
U(t)(x, y) = \int_0^\infty e^{it(r^2(y, x) - \theta^2)} \sum_{k=0}^\infty W_k(x, y) \theta^{d-\frac{3}{2} - k} d\theta \quad (t < \text{inj}(M, g))
\]
where \( U_o(x, y) = \Theta^{-\frac{1}{2}}(x, y) \) is the volume 1/2-density, where the higher coefficients are determined by transport equations, and where \( \theta^r \) is regularized at 0 (see below). This formula is only valid for times \( t < \text{inj}(M, g) \) but using the group property of \( U(t) \) it determines the wave kernel for all times. It shows that for fixed \( (x, t) \) the kernel \( U(t)(x, y) \) is singular along the distance sphere \( S_t(x) \) of radius \( t \) centered at \( x \), with singularities propagating along geodesics. It only represents the singularity and in the analytic case only converges in a neighborhood of the characteristic conoid.
One may use Duhamel’s formula to construct the exact solution as a Volterra series,

\[ U(t, x, y) = U_N(t, x, y) + \int_0^t U_N(t - s)(\partial_t^2 - \Delta)U_N(t - s)ds + \cdots. \]

Closely related but somewhat simpler is the even part of the wave kernel, \( \cos t\sqrt{\Delta} \) which solves the initial value problem

\[
\begin{aligned}
(\partial_t^2 - \Delta)u &= 0 \\
u|_{t=0} &= f \\
\partial_t u|_{t=0} &= 0
\end{aligned}
\]

Similar, the odd part of the wave kernel, \( \sin t\sqrt{\Delta} \) is the operator solving

\[
\begin{aligned}
(\partial_t^2 - \Delta)u &= 0 \\
u|_{t=0} &= 0 \\
\partial_t u|_{t=0} &= g
\end{aligned}
\]

These kernels only really involve \( \Delta \) and may be constructed by the Hadamard-Riesz parametrix method. As above they have the form

\[
\int_0^\infty e^{i\theta(r^2-t^2)} \sum_{j=0}^\infty W_j(x, y) \theta_{r^2}^{\frac{n-1}{2} - j} \theta d\theta \ \text{mod} \ C^\infty
\]

where \( W_j \) are the Hadamard-Riesz coefficients determined inductively by the transport equations

\[
\begin{aligned}
\frac{\Theta'}{2\Theta} W_0 + \frac{\partial W_0}{\partial r} &= 0 \\
4ir(x, y) \left\{ \left( \frac{k+1}{r(x, y)} + \frac{\Theta'}{2\Theta} \right) W_{k+1} + \frac{\partial W_{k+1}}{\partial r} \right\} &= \Delta_2 W_k.
\end{aligned}
\]

The solutions are given by:

\[
\begin{aligned}
W_0(x, y) &= \Theta^{-\frac{1}{2}}(x, y) \\
W_{j+1}(x, y) &= \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s)^{\frac{1}{2}} \Delta_2 W_j(x, x_s)ds
\end{aligned}
\]

where \( x_s \) is the geodesic from \( x \) to \( y \) parametrized proportionately to arc-length and where \( \Delta_2 \) operates in the second variable.

According to [GS], page 171,

\[
\int_0^\infty e^{i\theta\sigma} \theta^\lambda d\lambda = ie^{i\lambda\pi/2} \Gamma(\lambda + 1)(\sigma + i0)^{-\lambda-1}.
\]

One has,

\[
\int_0^\infty e^{i\theta(r^2-t^2)} \theta_{r^2}^{\frac{d-3}{2} - j} \theta d\theta = ie^{i(d+1-j)\pi/2} \Gamma(\frac{d-3}{2} - j + 1)(r^2 - t^2 + i0)^{j-\frac{d-3}{2} - 2}
\]

Here there is apparently trouble when \( d \) is odd since \( \Gamma(\frac{d-3}{2} - j + 1) \) has poles at the negative integers.

One then uses

\[
\Gamma(\alpha + 1 - k) = (-1)^k(1-\alpha)^{[\alpha]} \Gamma(\alpha + 1 - [\alpha]) \Gamma([\alpha] + 1 - \alpha) \frac{1}{\alpha + 1} \frac{1}{\alpha - [\alpha]} \Gamma(k - \alpha).
\]
We note that
\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}. \]

Here and above \( t^{-n} \) is the distribution defined by \( t^{-n} = \text{Re}(t + i0)^{-n} \) (see \[Be\], \[G.Sh., p.52,60\].) We recall that \( (t + i0)^{-n} = e^{-i\pi \frac{n}{2}} \frac{1}{\Gamma(n)} \int_0^\infty e^{itx}x^{-n-1}dx. \)

We also need that \( (x + i0)^{\lambda} \) is entire and
\[
(x + i0)^{\lambda} = \begin{cases} 
    e^{i\pi \lambda |x|^\lambda}, & x < 0 \\
    x^\lambda, & x > 0.
\end{cases}
\]

The imaginary part cancels the singularity of \( \frac{1}{\alpha - |x|} \) as \( \alpha \to \frac{d-3}{2} \) when \( d = 2m + 1 \). There is no singularity in even dimensions. In odd dimensions the real part is \( \cos \pi \lambda x^\lambda + x^\lambda \) and we always seem to have a pole in each term!

But in any dimension, the imaginary part is well-defined and we have
\[
\sin t\sqrt{\Delta}(x, y) = C_0 \text{sgn}(t) \sum_{j=0}^\infty (-1)^j w_j(x, y) \left( \frac{r^2 - t^2}{4j} \right)^{\frac{d-3}{2}-1} \mod C^\infty
\]

By taking the time derivative we also have,
\[
\cos t\sqrt{\Delta}(x, y) = C_0 |t| \sum_{j=0}^\infty (-1)^j w_j(x, y) \left( \frac{r^2 - t^2}{4j} \right)^{\frac{d-3}{2}-2} \mod C^\infty
\]

where \( C_0 \) is a universal constant and where \( W_j = \tilde{C}_0 e^{-ij \frac{\pi}{2}} 4^{-j} w_j(x, y) \). Similarly

12.3. Exact formula in spaces of constant curvature. The Poisson kernel of \( \mathbb{R}^{n+1} \) is the kernel of \( e^{-t\sqrt{\Delta}} \), given by
\[
K(t, x, y) = t^{-n}(1 + \frac{|x-y|^2}{t})^{-\frac{n+1}{2}} = t \left( t^2 + |x-y|^2 \right)^{-\frac{n+1}{2}}.
\]

It is defined only for \( t > 0 \), although formally it appears to be odd.

Thus, the kernel of \( e^{it\sqrt{\Delta}} \) is
\[
U(t, x, y) = i(t) \left( |x-y|^2 - t^2 \right)^{-\frac{n+1}{2}}.
\]

One would conjecture that the Poisson kernel of any Riemannian manifold would have the form
\[
K(t, x, y) = t \sum_{j=0}^\infty (t^2 + r(x, y)^2)^{-\frac{n+1}{2}+j} U_j(x, y)
\]

for suitable \( U_j \).
12.4. $\mathbb{S}^n$. One can determine the kernel of $e^{it\sqrt{\Delta}}$ on $\mathbb{S}^n$ from the Poisson kernel of the unit ball $B \subset \mathbb{R}^{n+1}$. We recall that the Poisson integral formula for the unit ball is:

$$u(x) = C_n \int_{\mathbb{S}^n} \frac{1 - |x'|^2}{|x - x'|^2} f(x') dA(x').$$

Write $x = r\omega$ with $|\omega| = 1$ to get:

$$P(r, \omega, \omega') = \frac{1 - r^2}{(1 - 2r\langle \omega, \omega' \rangle + r^2)^{\frac{n+1}{2}}}.$$ 

A second formula for $u(r\omega)$ is

$$u(r, \omega) = r^{A - \frac{n+1}{2}} f(\omega),$$

where $A = \sqrt{\Delta + \left(\frac{n-1}{4}\right)^2}$. This follows from by writing the equation $\Delta_{\mathbb{R}^{n+1}} u = 0$ as an Euler equation:

$$\{r^2\frac{\partial^2}{\partial r^2} + nr \frac{\partial}{\partial r} - \Delta_{\mathbb{S}^n}\} u = 0.$$ 

Therefore, the Poisson operator $e^{-tA}$ with $r = e^{-t}$ is given by

$$P(t, \omega, \omega') = C_n \frac{\sinh t}{\cosh t - \cos r(\omega, \omega')}^{\frac{n+1}{2}} - \frac{1}{(\cosh t - \cos r(\omega, \omega'))^{\frac{n+1}{2}}}.\frac{\cosh t}{\cosh t - \cos r(\omega, \omega')}^{\frac{n+1}{2}}.$$ 

Here, $r(\omega, \omega')$ is the distance between points of $\mathbb{S}^n$.

We analytically continue the expressions to $t > 0$ and obtain the wave kernel as a boundary value:

$$e^{itA} = \lim_{\epsilon \to 0^+} C_n i \sin t (\cosh \epsilon \cos t - i \sinh \epsilon \sin t - \cos r(\omega, \omega') - \frac{n+1}{2})$$

$$= \lim_{\epsilon \to 0^+} C_n i \sinh(i t - \epsilon)(\cosh(i t - \epsilon) - \cos r(\omega, \omega') - \frac{n+1}{2}).$$

If we formally put $\epsilon = 0$ we obtain:

$$e^{itA} = C_n i \sin t (\cos t - \cos r(\omega, \omega') - \frac{n+1}{2}).$$

This expression is singular when $\cos t = \cos r$. We note that $r \in [0, \pi]$ and that it is singular on the cut locus $r = \pi$. Also, $\cos : [0, \pi] \to [-1, 1]$ is decreasing, so the wave kernel is singular when $t = \pm r$ if $t \in [-\pi, \pi]$.

When $n$ is even, the expression appears to be pure imaginary but that is because we need to regularize it on the set $t = \pm r$. When $n$ is odd, the square root is real if $\cos t \geq \cos r$ and pure imaginary if $\cos t < \cos r$.

We see that the kernels of $\cos tA$, $\sin tA$, $\frac{\sin tA}{A}$ are supported inside the light cone $|r| \leq |t|$. On the other hand, $e^{itA}$ has no such support property (it has infinite propagation speed). On odd dimensional spheres, the kernels are supported on the distance sphere (sharp Huyghens phenomenon).

The Poisson kernel of the unit sphere is then

$$e^{-tA} = C_n \sinh t (\cosh t - \cos r(\omega, \omega'))^{-\frac{n+1}{2}}.$$
It is singular on the complex characteristic conoid when \( \cosh t - \cos r(\zeta, \zeta') = 0 \).

12.5. **Analytic continuation into the complex.** If we write out the eigenfunction expansions of \( \cos t \sqrt{\Delta}(x, y) \) and \( \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}}(x, y) \) for \( t = i\tau \), we would not expect convergence since the eigenvalues are now exponentially growing. Yet the majorants argument seems to indicate that these wave kernels admit an analytic continuation into a complex neighborhood of the complex characteristic conoid. Define the characteristic conoid in \( \mathbb{R} \times M \times M \) by \( r(x, y)^2 - t^2 = 0 \). For simplicity of visualization, assume \( x \) is fixed. Then analytically the conoid to \( \mathbb{C} \times M \times M \). By definition \( (\zeta, \bar{\zeta}, 2\tau) \) lies on the complexified conoid. That is, the series also converge after analytic continuation, again if \( r^2 - t^2 \) is small. If \( t = i\tau \) then we need \( r(\zeta, y)^2 + \tau^2 \) to be small, which either forces \( r(\zeta, y)^2 \) to be negative and close to \( \tau \) or else forces both \( \tau \) and \( r(\zeta, y) \) to be small.

If we wish to use orthogonality relations on \( M \) to sift out complexifications of eigenfunctions, then we need \( U(i\tau, \zeta, y) \) to be holomorphic in \( \zeta \) no matter how far it is from \( y \). So far, we do not have a proof that \( U(i\tau, \zeta, y) \) is globally holomorphic for \( \zeta \in M_y \) for every \( y \).

Regimes of analytic continuation. Let \( E(t, x, y) \) be any of the above kernels. Then analytically continue to \( E(t, \zeta, \bar{\zeta}) \) where \( r(\zeta, \zeta')^2 + \tau^2 \) is small. For instance if \( \zeta' = \zeta \) and \( \sqrt{\rho}(\zeta) = \frac{\tau}{2} \), then \( r(\zeta, \zeta')^2 + \tau^2 = 0 \).

If we analytically continue in \( \zeta \) and anti-analytically continue in \( \zeta' \), we seem to get a neighborhood of the conoid. We would like to analytically continue the Hadamard parametrix to a small neighborhood of the characteristic conoid. It is singular on the conoid.

12.6. **Hadamard-Riesz parametriz.** We try to construct the kernel as a homogeneous oscillatory integral

\[
E(t, x, y) = \int_0^\infty e^{i\theta(r^2 - t^2)} A(t, x, y, \theta) d\theta,
\]

where \( A \) is a polyhomogeneous symbol in \( \theta \),

\[
A(t, x, y, \theta) \sim \sum_{j=0}^\infty W_j(t, x, y) \theta^{\frac{n-3}{2} - j} d\theta \mod C^\infty
\]

The leading symbol order \( \theta^{\frac{n-3}{2}} \) is correct for \( \cos t \sqrt{\Delta} \). It should be \( \theta^{\frac{n-3}{2}} \) for \( \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}} \).

Recall that:

\[
\int_0^\infty e^{i\theta \lambda} \lambda d\lambda = i e^{i\lambda \pi/2} \Gamma(\lambda + 1)(\sigma + i0)^{-\lambda-1}.
\]

Hence

\[
\int_0^\infty e^{i\theta(r^2 - t^2)} \theta^{\frac{n-3}{2} - j} d\theta
\]
\[ = i e^{i \left( \frac{n-3}{2} - j \right) \pi / 2} \Gamma \left( \frac{n-3}{2} - j + 1 \right) \left( r^2 - t^2 + i0 \right)^{j-\frac{n-3}{2}} \]

When \( n \) is odd, \( \Gamma \left( \frac{n-3}{2} - j + 1 \right) \) has poles at the negative integers. Thus, this parametrix does not quite work on odd dimensional spaces (= even dimensional spacetimes).

- Riesz defined a holomorphic family of Riesz kernels \( (t - r^2)^\alpha \) and used an analytic continuation method to define the value when \( \alpha \) is a negative integer. He only studied the imaginary part, where there is no pole.
- Hadamard: see below.

\[ \sin \sqrt{\Delta}(x, y) = C_0 \text{sgn}(t) \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \left( r^2 - t^2 \right)^{j-\frac{n-3}{2} - 1} \frac{4 \Gamma(j - \frac{n-3}{2})}{\Gamma(\frac{n-3}{2})} \mod C^\infty \]

By taking the time derivative we also have,

\[ \cos t \sqrt{\Delta}(x, y) = C_0 |t| \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \left( r^2 - t^2 \right)^{j-\frac{n-3}{2} - 1} \frac{4 \Gamma(j - \frac{n-3}{2} - 1)}{\Gamma(\frac{n-3}{2})} \mod C^\infty \]

where \( C_0 \) is a universal constant and where the Hadamard-Riesz coefficients \( w_j(x, y) \) solve certain transport equations.

The coefficients \( W_j \) are determined inductively by the transport equations

\[ \frac{\Theta' r}{2\Theta} W_0 + \frac{\partial W_0}{\partial r} = 0 \]

\[ 4ir(x, y) \left\{ \left( \frac{k+1}{r(x, y)} + \frac{\Theta'}{2\Theta} \right) W_{k+1} + \frac{\partial W_{k+1}}{\partial r} \right\} = \Delta_y W_k. \]

The solutions are given by (HD167), i.e.

\[ W_0(x, y) = \Theta^{-\frac{1}{2}}(x, y) \]

\[ W_{j+1}(x, y) = \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s) \frac{1}{2} \Delta_2 W_j(x, x_s) ds \]

where \( x_s \) is the geodesic from \( x \) to \( y \) parametrized proportionately to arc-length and where \( \Delta_2 \) operates in the second variable.

13. Appendix: Lagrangian distributions and Fourier integral operators

13.1. Semi-classical Lagrangian distributions and Fourier integral operators. Semi-classical Fourier integral operators with large parameter \( \lambda = \frac{1}{\hbar} \) are operators whose Schwartz kernels are defined by semi-classical Lagrangian distributions,

\[ I_\lambda(x, y) = \int_{\mathbb{R}^N} e^{i\lambda \varphi(x, y, \theta)} a(\lambda, x, y, \theta) d\theta. \]

More generally, semi-classical Lagrangian distributions are defined by oscillatory integrals (see [HD]),

\[ u(x, \hbar) = \hbar^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar} \varphi(x, \theta)} a(x, \theta, \hbar) d\theta. \]
We assume that \( a(x, \theta, \hbar) \) is a semi-classical symbol,
\[
a(x, \theta, \hbar) \sim \sum_{k=0}^{\infty} \hbar^{\mu+k} a_k(x, \theta).
\]

The critical set of the phase is given by
\[
C_\varphi = \{(x, \theta) : d_\theta \varphi = 0\}.
\]
The phase is called non-degenerate if
\[
d(\frac{\partial \varphi}{\partial \theta_1}), \ldots, d(\frac{\partial \varphi}{\partial \theta_N})
\]
are independent on \( C_\varphi \). Thus, the map
\[
\varphi'_\theta := \left( \frac{\partial \varphi}{\partial \theta_1}, \ldots, \frac{\partial \varphi}{\partial \theta_N} \right) : X \times \mathbb{R}^N \rightarrow \mathbb{R}^N
\]
is locally a submersion near 0 and \((\varphi'_\theta)^{-1}(0)\) is a manifold of codimension \( N \) whose tangent space is \( \ker D\varphi_\theta \). Then
\[
T_{(x, \theta)}C_\varphi = \ker d_x d_\theta \varphi.
\]

We write a tangent vector to \( M \times \mathbb{R}^N \) as \((\delta x, \delta \theta)\). The kernel of
\[
D\varphi'_\theta = \begin{pmatrix} \varphi''_{\theta x} & \varphi''_{\theta \theta} \end{pmatrix}
\]
is \( T_{(x, \theta)}C_\varphi \). I.e. \((\delta x, \delta \theta) \in TC_\varphi \) if and only if \( \varphi''_{\theta x} \delta x + \varphi''_{\theta \theta} \delta \theta = 0 \). Indeed, \( \varphi'_\theta \) is the defining function of \( C_\varphi \) and \( d\varphi_\theta \) is the defining function of \( TC_\varphi \). From [HoIV] Definition 21.2.5: The number of linearly independent differentials \( \frac{d\varphi}{d\theta} \) at a point of \( C_\varphi \) is \( N - e \) where \( e \) is the excess. Then \( C \rightarrow \Lambda \) is locally a fibration with fibers of dimension \( e \). So to find the excess we need to compute the rank of \( \left( \varphi''_{x \theta}, \varphi''_{\theta \theta} \right) \) on \( T_{x, \theta}(\mathbb{R}^N \times M) \).

Non-degeneracy is thus the condition that
\[
\begin{pmatrix} \varphi''_{\theta x} & \varphi''_{\theta \theta} \end{pmatrix} \text{ is surjective on } C_\varphi \quad \iff \quad \begin{pmatrix} \varphi''_{x \theta} \\ \varphi''_{\theta \theta} \end{pmatrix} \text{ is injective on } C_\varphi.
\]
If \( \varphi \) is non-degenerate, then \( \iota_\varphi(x, \theta) = (x, \varphi'_{x}(x, \theta)) \) is an immersion from \( C_\varphi \rightarrow T^*X \). Note that
\[
d_{\iota_\varphi}(\delta x, \delta \theta) = (\delta x, \varphi''_{xx} \delta x + \varphi''_{x \theta} \delta \theta).
\]
So if \( \begin{pmatrix} \varphi''_{x \theta} \\ \varphi''_{\theta \theta} \end{pmatrix} \) is injective, then \( \delta \theta = 0 \).

If \((\lambda_1, \ldots, \lambda_n)\) are any local coordinates on \( C_\varphi \), extended as smooth functions in neighborhood, the delta-function on \( C_\varphi \) is defined by
\[
d_{C_\varphi} := \frac{|d\lambda|}{|D(\lambda, \varphi'/\theta)/D(x, \theta)|} = \frac{d\text{vol}_{T_{\lambda, \varphi}/M} \otimes d\text{vol}_{\mathbb{R}^N}}{d\varphi_{x}^{\theta} \wedge \cdots \wedge d\varphi_{\theta}^{\varphi_{x}}},
\]
where the denominator can be regarded as the pullback of \( d\text{Vol}_{\mathbb{R}^N} \) under the map
\[
d_{\theta, x} d_\theta \varphi(x_0, \theta_0).
\]
The symbol $\sigma(\nu)$ of a Lagrangian (Fourier integral) distributions is a section of the bundle $\Omega^1_2 \otimes M^1_2$ of the bundle of half-densities (tensor the Maslov line bundle). In terms of a Fourier integral representation it is the square root $\sqrt{dC_\varphi}$ of the delta-function on $C_\varphi$ defined by $\delta(d_\theta \varphi)$, transported to its image in $T^* M$ under $\iota_\varphi$.

**Definition:** The principal symbol $\sigma_u(x_0, \xi_0)$ is

$$ \sigma_u(x_0, \xi_0) = a_0(x_0, \xi_0) \sqrt{dC_\varphi}. $$

It is a $\frac{1}{2}$ density on $T(x_0, \xi_0) \Lambda_\varphi$ which depends on the choice of a density on $T_{x_0} M$).

### 13.2. Homogeneous Fourier integral operators.

A homogeneous Fourier integral operator $A : C^\infty(X) \to C^\infty(Y)$ is an operator whose Schwartz kernel may be represented by an oscillatory integral

$$ K_A(x, y) = \int_{\mathbb{R}^N} e^{i\varphi(x, y, \theta)} a(x, y, \theta) d\theta $$

where the phase $\varphi$ is homogeneous of degree one in $\theta$. We assume $a(x, y, \theta)$ is a zeroth order classical polyhomogeneous symbol with $a \sim \sum_{j=0}^\infty a_j$, $a_j$ homogeneous of degree $-j$.

We refer to [DS], [GS2] and especially to [HoIV] for background on Fourier integral operators. We use the notation $I^m(X \times Y, C)$ for the class of Fourier integral operators of order $m$ with wave front set along the canonical relation $C$, and $WF'(F)$ to denote the canonical relation of a Fourier integral operator $F$.

When

$$ \iota_\varphi : C_\varphi \to \Lambda_\varphi \subset T^*(X, Y), \ \ i_\varphi(x, y, \theta) = (x, d_x \varphi, y, -d_y \varphi) $$

is an embedding, or at least an immersion, the phase is called non-degenerate. Less restrictive, although still an ideal situation, is where the phase is clean. This means that the map $\iota_\varphi : C_\varphi \to \Lambda_\varphi$, where $\Lambda_\varphi$ is the image of $\iota_\varphi$, is locally a fibration with fibers of dimension $e$. From [HoIV] Definition 21.2.5, the number of linearly independent differentials $d^\varphi_{\partial \theta}$ at a point of $C_\varphi$ is $N - e$ where $e$ is the excess.

We a recall that the order of $F : L^2(X) \to L^2(Y)$ in the non-degenerate case is given in terms of a local oscillatory integral formula by $m + \frac{N}{2} - \frac{n}{4}$, where $n = \dim X + \dim Y$, where $m$ is the order of the amplitude, and $N$ is the number of phase variables in the local Fourier integral representation (see [HoIV], Proposition 25.1.5); in the general clean case with excess $e$, the order goes up by $\frac{e}{2}$ ([HoIV], Proposition 25.1.5'). Further, under clean composition of operators of orders $m_1$, $m_2$, the order of the composition is $m_1 + m_2 - \frac{e}{2}$ where $e$ is the so-called excess (the fiber dimension of the composition); see [HoIV], Theorem 25.2.2.

The definition of the principal symbol is essentially the same as in the semi-classical case. As discussed in [SV], (see (2.1.2) and ((2.2.5) and Definition 2.7.1)), if an oscillatory integral is represented as

$$ I_{\varphi, a}(t, x, y) = \int e^{i\varphi(t, x, y, \eta)} a(t, x, y, \eta) \zeta(t, x, y, \eta) d_\varphi(t, x, y, \eta) d\eta, $$

where

$$ d_\varphi(t, x, y, \eta) = | \det \varphi_{x, \eta} |^{\frac{1}{2}} $$
and where the number of phase variables equals the number of $x$ variables, then the phase is non-degenerate if and only if $(\varphi_{x\eta}(t, x, y, \eta)$ is non-singular. Then $a_0|\det \varphi_{x,\eta}|^{-\frac{1}{2}}$ is the symbol.

The behavior of symbols under pushforwards and pullbacks of Lagrangian submanifolds are described in [GSt1], Chapter IV. 5 (page 345). The main statement (Theorem 5.1, loc. cit.) states that the symbol map $\sigma : I^m(X, \Lambda) \to S^m(\Lambda)$ has the following pullback-pushforward properties under maps $f : X \to Y$ satisfying appropriate transversality conditions,

\[
\begin{align*}
\sigma(f^*\nu) &= f^*\sigma(\nu), \\
\sigma(f_*\mu) &= f_*\sigma(\mu),
\end{align*}
\]

Here, $f_*\sigma(\mu)$ is integration over the fibers of $f$ when $f$ is a submersion. In order to define a pushforward, $f$ must be a “morphism” in the language of [GSt2], i.e. must be accompanied by a map $r(x) : \bigwedge^{\frac{1}{2}} TY_{f(x)} \to \bigwedge^{\frac{1}{2}} TX_x$, or equivalently a half-density on $N^*(\text{graph}(f))$, the co-normal bundle to the graph of $f$ which is constant long the fibers of $N^*(\text{graph}(f)) \to \text{graph}(f)$.

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