Counting equilibria in complex systems via random matrices

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Abstract

How many equilibria will a large complex system, modeled by $N$ randomly coupled autonomous nonlinear differential equations typically have? How many of those equilibria are stable, that is are local attractors of the nearby trajectories? These questions arise in many applications and can be partly answered by employing the methods of Random Matrix Theory. The lectures will outline these recent developments.
LECTURE 1

May Model of a Complex System

Will diversity make a food chain more or less stable? The prevailing view in the mid-twentieth century was that diverse ecosystems have greater resilience to recover from events displacing the system from equilibrium and hence are more stable. This ‘ecological intuition’ was challenged by Robert May in 1972 [1]. At that time, computer simulations suggested that large complex systems assembled at random might become unstable as the system complexity increases [2]. May’s 1972 paper complemented that work with an analytic investigation of the neighbourhood stability of a model ecosystem whereby $N$ species at equilibrium are subject to random interactions.

The time evolution of large complex systems, of which model ecosystems is one example, is often described within the general mathematical framework of coupled first-order nonlinear ordinary differential equations (ODEs).

\begin{equation}
\frac{dx}{dt} = F(x), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}_N
\end{equation}

The choice of the vector field $F(x) = (F_1(x), \ldots, F_N(x))$ and hence detailed properties of the phase space trajectories strongly depend on specific area of applications and vary considerably from model to model. In populational ecology a basic model is the multispecies Lotka-Volterra system [3] for population densities $x_i \geq 0$, with the choice $F_i = x_i(r_i + \sum_{j=1}^{N} J_{ij}x_j)$ where $r_i$ is the intrinsic growth/death rate of the population (for $r_i > 0$ and $r_i < 0$, respectively), and the parameters $J_{ij}$ model the influence of growth in population $j$ on growth rate in population $i$. In that setting $J_{ii}$ are usually negative to ensure a self-regulation mechanism, whereas $J_{i\neq j}$ can be of any sign, reflecting either competitive (with $J_{i\neq j} > 0$) or predator-prey (with $J_{i\neq j} > 0$ and $J_{ji} < 0$) relations between the leaving organisms.

Another area where equation (1) was most popular and well-studied numerically is theory of neural network consisting of randomly interconnected neural units [4], with the particular choice of $F_i = -x_i + \sum_{j=1}^{n} J_{ij}S(x_j)$ where $S(x)$ is an odd sigmoid function (i.e. $S(-x) = -S(x)$, $S'(x) \geq 0$, $S(\infty) < \infty$) representing the synaptic nonlinearity and $J_{ij}$ are representing the synaptic connectivity between neuron $i$ and $j$ and can be of any sign: excitatory or inhibitory. Other examples of this sort include machine learning [7], complex gene regulatory networks [5, 6], and catalytic reaction networks [8].

Study of any autonomous dynamical system (1.1) traditionally starts with the “local stability analysis” amounting to determining all possible points of equilibria defined as zeroes of the vector field: $F(x) = 0$ and analysing the dynamical behaviour in the vicinity of each those points by Taylor-expanding and subsequently replacing the non-linear interaction functions near the equilibrium with their linear approximations. In the context of a generic system, the Hartman-Grobner theorem
then asserts that the neighbourhood stability of a typical equilibrium is governed by its linear approximation. It is along these lines that May suggested to look at the model linear system, thinking of \( y_i \) as the deviations from a local equilibrium:

\[
\frac{dy_j}{dt} = -\mu y_j + \sum_{k=1}^{N} J_{jk} \, y_k, \quad j = 1, \ldots, N, \tag{1.2}
\]

to shed some light on stability of large complex nonlinear systems. Here \( J = (J_{jk}) \) is the coupling matrix and \( \mu > 0 \) to ensure that in the absence of interactions, i.e., when all \( J_{jk} = 0 \), the system (1.2) is self-regulating: if disturbed from the equilibrium \( y_1 = y_2 = \ldots = y_N = 0 \) it returns back with some characteristic relaxation time set by \( \mu \). In an ecological context \( y_j(t) \) is interpreted as the variation about the equilibrium value, \( y_j = 0 \), in the population density of species \( j \) at time \( t \). The element \( J_{jk} \) of the coupling matrix \( J \), which is known as the community matrix in ecology, measures the per capita effect of species \( k \) on species \( j \) at the presumed equilibrium. Generically, the community matrix is asymmetric, \( J_{jk} \neq J_{kj} \). The dynamics of linear model \( y(t) = e^{-\mu t + J} y(0) \) is obviously controlled by spectrum of the matrix \( J \). According to Hartman-Grobner theorem the equilibrium at \( y = 0 \) becomes unstable iff the matrix \( -\mu I + J \) has eigenvalues with positive real part.

In reality, the detailed description of the vector fields \( F(\mathbf{x}) \) is rarely available for large enough systems of considerable complexity in problems of practical interest. At the same time it is natural to assume that in order to understand generic qualitative properties of the global dynamics of large systems of ODE’s shared by many models of similar type it may be enough to retain only a few characteristic structural features of the vector field, treating the rest as random. In part such approach is methodologically inspired by undisputed success of the Random Matrix Theory (RMT) which manages to describe many properties of systems of very diverse nature, such as energy levels of heavy nuclei, zeroes of the Riemann zeta-function and distances between tightly parked cars in a single conceptual framework \[10\].

Along these lines May considered an ensemble of community matrices \( J \) assembled at random, whereby the matrix elements \( J_{jk} \) are sampled from a probability distribution, which for simplicity may be assumed Gaussian, with zero mean and a prescribed variance \( \alpha^2 \). The corresponding matrix \( J \) is said to belongs to the real Ginibre ensemble which we will denote GinOE (see below). Invoking early studies on eigenvalues of GinOE by Ginibre\[11\] May claimed that for large \( N \) the largest real part of the eigenvalues of \( J \) is typically \( \alpha \sqrt{N} \). Obviously, the model’s stability is then controlled by the ratio \( m = \mu/(\alpha \sqrt{N}) \). For \( N \) large, the system (1.2) will almost certainly be stable if \( m > 1 \) and unstable if \( m < 1 \), with a sharp transition between the two types of behaviour with changing either \( \mu, \alpha \) or \( N \). In particular, for fixed \( \mu, \alpha \) the system (1.2) will almost certainly become unstable for \( N \) sufficiently large.

Despite the simplistic character of May’s model, his pioneering work gave rise to a long standing ‘stability versus diversity’ debate, which is not fully settled yet, see e.g. \[12\], and played a fundamental role in theoretical ecology by prompting ecologists to think about special features of real multi-species ecosystems that help such systems to remain stable. Variations of May’s model are still being discussed nowadays in the context of neighbourhood stability, see \[12, 13\] and references therein. Moreover, May’s stability analysis in fact became a paradigm in complex
systems of diverse nature; for example, it was used in attempts to understand behaviour of financial ecosystems [14, 15].

One obvious limitation of the neighbourhood stability analysis is that it gives no insight into the model behaviour beyond the instability threshold. Hence May’s model has only limited bearing on the dynamics of populations operating out-of-equilibrium. An instability does not necessarily imply lack of persistence: populations could coexist thanks to limit cycles or chaotic attractors, which typically originate from unstable equilibrium points. Important questions posing serious challenge then relate to classification of equilibria by stability, studying their basins of attraction, and other features of global dynamics [12].

In the recent papers [16, 17] a simple nonlinear extension of the May model was proposed by retaining only the bare essentials - nonlinearity and stochasticity. Much in the spirit of the May’s original approach, the proposed model was simple enough to allow for an analytic treatment yet at the same time rich enough to exhibit a non-trivial behaviour. In particular, it captures an instability transition of the May-Wigner type, but now on the global scale. It also sheds additional light on the nature of this transition by relating it to an exponential explosion in the number of unstable equilibria. Interestingly, despite the nonlinear setting of the problem the properties of Ginibre-like random matrices again play a central role in its analysis.

The goal of these lectures to give a quite detailed account of the theory developed in [16, 17]. For this one needs first to develop the necessary background knowledge on eigenvalues of Ginibre Ensemble, for the standard case of unit variance $\alpha = 1$, and we will denote the corresponding matrixes by $G$. Next chapter will
aim exactly at this, using, in particular, the review paper [18] for the background material.

**Real Ginibre Ensemble: a concise review**

We consider $N \times N$ square matrices $G \in \mathcal{M}_N(\mathbb{R})$ with independent identically distributed matrix elements

$$G_{j,k} \sim \mathcal{N}(0,1).$$

For this ensemble we will use a notation GinOE, underlying orthogonal symmetry of the distribution. We will also denote by the angular brackets $\langle \cdots \rangle_{\text{GinOE}}$ the expectation of any function $F : \mathbb{R}^{N \times N} \to \mathbb{C}$ with respect the associated probability distribution, and will frequently omit the corresponding subscript.

**Remark 1.1.** One can also introduce in a similar way the so called complex Ginibre ensemble (GinUE) defined by

$$G_{j,k} = g_{j,k}^{(1)} + i \cdot g_{j,k}^{(2)}, \text{ with i.i.d. } g_{j,k}^{(c)} \sim \mathcal{N}(0,1/2),$$

as well as the so-called quaternion (GinSE) ensemble which is largely immaterial for our purposes.

Assigning the Dyson’s index $\beta = 1,2,4$ to GinOE, GinUE and GinSE, respectively one can write for all three ensembles the Joint Probability Density (JPD) with respect to flat Lebesgue measure in the form

$$P_{\beta}(G) = \left(\frac{\beta}{2\pi}\right)^{\frac{N^2}{2}} \exp\left\{-\frac{\beta}{2} \text{Tr} GG^*\right\}.$$

Let $z_1, \ldots, z_N$ be eigenvalues of $G \in \text{GinOE}$. We always assume that eigenvalues are ordered in the following way: real evs in descending order are followed by pairs of conjugate complex eigenvalues, where pairs are ordered in lexicographical order. We may assume that there is no multiple eigenvalues because of:

**Lemma 1.1.** For every $N$, the set of elements of $\mathcal{M}_N(\mathbb{R})$ with multiple eigenvalues has zero Lebesgue measure in $\mathbb{R}^{N \times N}$ (see Exercises).

Let us introduce sets

$$\mathcal{X}_{L,M} = \{ G \in \mathbb{R}^{N \times N} : |\text{spec}(G) \cap \mathbb{R}| = L, |\text{spec}(G) \cap \mathbb{C}_+| = M \}.$$

i.e. $G$ has exactly $L$ real eigenvalues and $M$ pairs of complex conjugate eigenvalues. Then

$$\mathcal{M}_N(\mathbb{R}) = \bigsqcup_{L+2M=N} \mathcal{X}_{L,M}.$$

Below we study eigenvalues distribution for every set $\mathcal{X}_{L,M}$ independently. Let us fix $L$ and $M$. For every $G \in \mathcal{X}_{L,M}$ we write its eigenvalues as

$$(\lambda_1, \ldots, \lambda_L, x_1 + iy_1, x_1 - iy_1, \ldots, x_M + iy_M, x_M - iy_M)$$

with $\lambda_1 > \lambda_2 > \ldots > \lambda_L$ and $x_1 \geq x_2 \geq \ldots \geq x_M$ and $y_j > 0$, $j = 1, M$. 

Theorem 1.1 (see, [19],[20], Exercises). Let \((z_1, z_2, \ldots, z_N)\) be the ordered set of either real or pairs of complex conjugate eigenvalues of matrix \(G\) taken randomly from the Real Ginibre Ensemble. Then eigenvalues Joint Probability Density (conditional) function is given by

\[
P_1^{(L,M)}(z_1, \ldots, z_N) = 2^M \left(2^{N(N+1)/4} \prod_{j=1}^{N} \Gamma(j/2)\right)^{-1} \prod_{1 \leq j < k \leq N} |z_j - z_k| \exp \left\{-\frac{1}{2} \sum_{1 \leq j \leq N} |z_j|^2 \right\} \prod_{1 \leq j \leq N} \sqrt{\text{erfc} \left(\sqrt{2} |\text{Im}z_j|\right)} 1_{G \in \mathcal{X}_{L,M}}.
\]

where \(\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt\).

Remark 1.2 (see, [11]). Similarly to the above one can derive eigenvalues JPDF in the case of GinUE (and eventually GinSE) ensembles. Corresponding density for GinUE is given below

\[
P_2(z_1, \ldots, z_N) = \frac{1}{(2\pi)^N \prod_{k=1}^{N-1} \Gamma(k/2)} \prod_{j < k} |z_j - z_k|^2 \exp \left\{-\sum_{j} |z_j|^2 \right\},
\]

showing that matrices with real eigenvalues in this case have zero Lebesgue measure.

The most important property of GinOE is that averages for a class of multi-

Exercise 1.1 (see, e.g. [21],[22], Exercises). Let \(N\) be even and \(f : \mathbb{C} \to \mathbb{C}\) be an arbitrary (up to some integrability conditions) function of complex argument. Define \(F : \mathbb{R}^{N \times N} \to \mathbb{C}\) by an identity

\[
F(G) = \prod_{1 \leq j \leq N} f(z_j).
\]

Let \(\{P_{j-1}(z)\}_{j=1}^{N}\) be a set of monic polynomials in \(z\) of degree \(j - 1\). Then

\[
\langle F \rangle_{\text{GinOE}} = \left(2^{N(N+1)/4} \prod_{j=1}^{N} \Gamma(j/2)\right)^{-1} \text{Pf}(U),
\]

where \(U\) is an \(N \times N\) skew-symmetric matrix with

\[
U_{j,k} = \int_{\mathbb{C}^2} dz_1 dz_2 f(z_1) f(z_2) \mathcal{F}(z_1, z_2) P_{j-1}(z_1) P_{j-1}(z_2)
\]

and the skew-symmetric measure

\[
\mathcal{F}(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) = e^{-(z_1^2 + z_2^2)/2} \left[2i \delta^2(z_1 - z_2) \text{sgn}(y_1) \text{erfc} \left(|y_1| \sqrt{2}\right) + \delta(y_1) \delta(y_2) \text{sgn}(x_2 - x_1)\right].
\]

where we understand \(\delta^2(z - u)\) as \(\delta(\text{Re}z - \text{Re}u)\delta(\text{Im}z - \text{Im}u)\) for some fixed \(u \in \mathbb{C}\).

Remark 1.3. In the case of odd \(N\) one should take \(U\) to be skew symmetric matrix of size \(N + 1 \times N + 1\) with the last column equal to

\[
U_{j,N+1} = \int_{\mathbb{R}} f(t) P_{j-1}(t) e^{-t^2/2}, j = 1, \ldots, N.
\]
Proposition 1.1. For any skew symmetric matrix $A$ of size $2n \times 2n$ we write $\text{Pf} A$ to denote its Pfaffian given by

$$
\text{Pf} A = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{j=1}^{n} A_{\sigma(2j-1), \sigma(2j)},
$$

where the summation is taken over all permutations of $(1,2,\ldots,2n)$.

Definition 1.1. For any skew symmetric matrix $A$ of size $2n \times 2n$ one has $\text{Pf}^2 A = \det A$, (see Exercises).

Let $\mathcal{P}(z_1, \ldots, z_N)$ stand for the unconditional JPDF given by the sum of all conditional JPDFs (1.3). Introduce the $k$–point correlation functions

$$
R_k(z_1, z_2, \ldots, z_k) = \frac{N!}{(N-k)!} \int_{C^{N-k}} \mathcal{P}(z_1, \ldots, z_N) \prod_{j=k+1}^{N} d^2 z_j
$$

In particular, $R_1(z) = N \int_{C} \mathcal{P}(z_1, \ldots, z_N) d^2 z_2, \ldots, d^2 z_N$ is proportional to the mean eigenvalue density $\langle \rho(z) \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta^2 (z - z_i) \right\rangle$.

It turns out that Thm (1.2) eventually provides an efficient tool of calculating all possible correlation functions. For example, consider

$$
f(z) = 1 + \delta^2(z - u),
$$

for some fixed $u \in \mathbb{C}$. Then the corresponding average

$$
\langle F \rangle_{\text{GinOE}} = \left\langle \prod_{j=1}^{N} (1 + \delta^2(z_j - u)) \right\rangle_{\text{GinOE}} = 1 + N \langle \rho(u) \rangle.
$$

since all averages containing products of $\delta$ -functions will be zero, because they correspond to matrices with an eigenvalue of multiplicity greater than one.

In fact, by using Theorem 1.2 with judiciously chosen functions $f$ one is able to calculate in a similar way all the correlation functions explicitly.

Theorem 1.3 (see [22],[23]). Define the kernel

$$
K_N(z_1, z_2) = \sum_{k=1}^{N} \sum_{l=1}^{N} [U^{-1}]_{kl} P_{k-1}(z_1) P_{l-1}(z_2)
$$

where the antisymmetric matrix $U$ is defined in (1.5) with the following choice: $f(z) = 1 + \sum_{j=1}^{n} \delta^2(z - z_j)$. Then

$$
R_n(z_1, \ldots, z_n) = \text{Pf} (Q_{kl}), \quad Q_{kl} = \begin{pmatrix} K_{kl} & G_{kl} \\ -G_{lk} & W_{kl} \end{pmatrix}, \quad k, l = 1, 2, \ldots, n
$$

that is the Pfaffian of $2n \times 2n$ matrix built of the $n^2$ matrices $Q_{kl}$ whose entries are expressed in terms of the kernel (1.12) and the skew-symmetric measure (1.6) as

$$
K_{kl} = K_N(z_k, z_l), \quad G_{k,l} = \int_C K_N(z_k, z) F(z, z_l) d^2 z,
$$

$$
W_{k,l} = -F(z_k, z_l) + \int_{C^2} F(z_k, z) K_N(z, z') F(z', z_l) d^2 z d^2 z'
$$

and symmetries $K_{kl} = -K_{lk}$, $W_{kl} = -W_{lk}$. See Exercises.
For example, the one-point correlation function (proportional to the mean density) is given by

\[ R_1(z_1) = \int_C K_N(z_1, z) \mathcal{F}(z, z_1) \, d^2z = R_1^C(z_1) + \delta(y) R_1(x_1) \]

in terms of the densities of complex and purely real eigenvalues, respectively. In particular, in view of (1.6)

\[ R_1^C(z) = K_N(z, \pi) \cdot 2e^{-\frac{z_2^2}{2}} \text{erfc} (|y| \sqrt{2}) \text{sgn}(y) \]

The kernel \( K_N(z_1, z_2) \) can be found explicitly with a judiciously chosen set of monic polynomials in (1.8), or by alternative tricks \[18\] with the result being rather simple, see **Exercises**:

\[ K_N(z_1, z_2) = \frac{z_1 - z_2}{2\sqrt{2\pi}} e^{z_1 z_2} \frac{\Gamma(N - 1, z_1 z_2)}{(N - 2)!}, \]

where we used the incomplete \( \gamma \)-function defined as

\[ \Gamma(N, a) = (N - 1)! e^{-a} \sum_{n=0}^{N-1} \frac{a^n}{n!} = \int_a^\infty e^{-t} t^{N-1} \, dt. \]

This immediately gives the mean density of complex eigenvalues (1.11) for \( z = x + iy \)

\[ R_1^{(c)}(x, y) = \frac{2|y| e^{2y^2} \text{erfc}(\sqrt{2}|y|)}{\sqrt{2\pi}} \frac{\Gamma(N - 1, x^2 + y^2)}{(N - 2)!} \]

and after some effort the density of real eigenvalues in the form (see **Exercises**)

\[ R_1^{(r)}(x) = \frac{\Gamma(N - 1, x^2)}{\sqrt{2\pi}(N - 2)!} + \frac{x^{N-1}}{\sqrt{2\pi}(N - 2)!} e^{-x^2/2} \int_0^x e^{-\frac{u^2}{2}} u^{N-2} \, du \]
Large–N asymptotics and Large Deviations for Ginibre ensemble

For the goals of the present Lectures, that is for understanding stability of a large complex system, we are mostly interested in understanding the behaviour of the eigenvalues of $G$ in the limit of large $N \gg 1$, especially in the right-most eigenvalues. In every realization of the Ginibre Ensemble the positions of its eigenvalues are traditionally characterized by the empirical spectral counting measure $\mu_N(z)$ such that $d\mu_N = \rho_N(z)d^2z$ where $d^2z = d(\text{Re}z)\,d(\text{Im}z) = dx\,dy$ and the density of the empirical measure is given by $\rho_N(z) = \frac{1}{N}\sum_{i=1}^{N}\delta^2(z - z_i)$. In particular, it is easy to see the mean value of such density in the complex plane $R_{1}(\rho)\equiv \langle \rho_N(z)\rangle$ given for finite $N$ by (1.14) is described asymptotically for $N \to \infty$ by the so-called 'circular law' (2.1)

$$R_{1}^{(eq)}(z) = \begin{cases} \frac{1}{\pi} & |z| < \sqrt{N} \\ 0 & \text{otherwise} \end{cases}$$

To see this one can use the easily verifiable identity (see Exercises):

$$\lim_{N \to \infty} \frac{\Gamma(N-1,Na)}{(N-2)!} = 1, \text{ if } 0 \leq a < 1 \text{ and } 0 \text{ if } a > 1$$

and the form (2.4) indeed emerges from (1.14) by rescaling $x = \tilde{x}\sqrt{N}$ and $y = \tilde{y}\sqrt{N}$ with finite $\tilde{x} < \infty, \tilde{y} < \infty$ and using the large-$N$ asymptotics

$$\text{erfc} \left( \sqrt{2}\tilde{y}|\sqrt{N} \right) \approx \sqrt{\frac{1}{2\pi N}} \frac{1}{|\tilde{y}|} e^{-2N\tilde{y}^2}.$$ 

One also finds asymptotically a constant density of real eigenvalues inside the same circle: $R_{1}^{(eq)}(x) = \frac{1}{\sqrt{2\pi}}$ for $|x| < \sqrt{N}$ and zero outside. This implies that for large $N \gg 1$ a typical Ginibre matrix has of the order of $O(\sqrt{N})$ real eigenvalues uniformly filling in the interval $|x| < \sqrt{N}$, whereas of the order of $O(N)$ eigenvalues uniformly fill in the interior of the circle $|x| \leq \sqrt{N}$. Coming back to the issue of the right-most eigenvalue, we see that the mean value of $x_m = \max(\text{Re}z_i)$ should be given by the end of support for the mean density, so that $\langle x_m \rangle \approx \sqrt{N}$. In fact, it is well known that not only the mean density, but the whole measure $\mu_N(z)$ weakly converges to the non-random 'equilibrium measure' $\mu_{eq}(z)$ with the density (2.1), so in a certain sense deviations of $x_m$ from $\sqrt{N}$ should be small, which was precisely the basis for May’s conclusions on the threshold of linear stability. To characterize more precisely how much the empirical measure $\mu_N(z)$ deviates from

\footnote{The latter statement is correct if we are not interested in what is going on close to the real line, for $|y| \sim 1 \ll \sqrt{N}$ where inside the circle there is a certain depletion of complex eigenvalues: $R_{1}^{(eq)}(z) \approx \sqrt{2\pi}|y|e^{3y^2} \text{erfc} (\sqrt{2}|y|)$.}
\[ \mu_{eq}(z) \text{ and } x_m \text{ from its expectation for large } N \gg 1 \text{ is possible in the framework of the so-called Large Deviation Theory (LDT) developed for GinOE in [24].} \]

Before discussing the LDT results we need to make the following remark. Eventually, it is more convenient to work with the rescaled version of the Ginibre matrices \( Z = G/\sqrt{N} \) for which eigenvalues in large \( N \gg 1 \) limit fill in the interior of the unit circle, with the equilibrium mean density given by

\[ \rho_{eq}(z) = \begin{cases} \frac{1}{\pi}, & |z| < 1 \\ 0, & \text{otherwise} \end{cases} \]

For such an arrangement of eigenvalues an average separation between the eigenvalues is of the order of \( N^{-1/2} \). From now on we will assume such a rescaling done.

It is natural to expect that nontrivial correlations between different eigenvalues \( z_1, \ldots, z_N \) of the rescaled matrices \( Z \) exist only on the scale \( |z_i - z_j| \sim N^{-1/2} \). In particular, one may expect that if the values of the variables \( z_1 \neq z_2 \ldots \neq z_k \) are distinct and fixed, in the limit \( N \to \infty \) one should have asymptotic factorization of the corresponding higher correlation functions defined in (1.7):

\[ R_k(z_1, z_2, \ldots, z_k) \sim \prod_{i=1}^{k} R_1(z_i) \]

This fact can be verified by employing the Thm. 1.3 (see Exercises) and later on will be used when developing the LDT for the right-most eigenvalue of \( Z \).

Let us denote \( E_\mu \) the space of all normalized probability measures \( \mu \) symmetric (with respect to complex conjugation) equipped with weak convergence. Consider any (measurable open) subset \( B \in E_\mu \) and let \( \mu_N \) be the normalized eigenvalue counting measure for the rescaled Ginibre ensemble \( Z \) on the complex plane \( \mathbb{C} \). Then for large \( N \gg 1 \) the probability \( P_N(B) = \text{Prob}[\mu_N \in B] \) can be shown to behave as

\[ P_N(B) \approx \exp \left\{ -N^2 \inf_B \mathcal{J}[\mu_N] \right\}, \]

with the LDT rate functional \( \mathcal{J}[\mu_N] \) given by

\[ \mathcal{J}(\mu) = \frac{1}{2} \int_{\mathbb{C}} |z|^2 d\mu(z) - \frac{1}{2} \int_{\mathbb{C}^2} \log |z - w| d\mu(z) d\mu(w) - \frac{3}{8} \]

One can further show that the above functional is nonnegative: \( \mathcal{J}[\mu] \geq 0 \), and has a unique minimizer. Moreover, it takes its minimal value zero precisely at the equilibrium measure \( \mu(z) = \mu_{eq}(z) \) from (2.4), that is \( \mathcal{J}[\mu_{eq}] = 0 \). For the precise meaning and proof of these statements see [24].

The above formulæ show that deviations from the equilibrium density are exponentially penalized with the rate proportional to \( N^2 \). One can informally understand the origin of (2.7) by noting that the JPD of complex eigenvalues for its counterpart GinUE, given in Eq.(1.4) can be, after the rescaling \( z_i \to \sqrt{N}z_i \), informally rewritten as

\[ P_2(z_1, \ldots, z_N) \propto \exp \left\{ -N^2 \left[ \int_{\mathbb{C}} |z|^2 d\mu_N(z) - \frac{1}{2} \int_{\mathbb{C}^2} \ln |z - w| d\mu_N(z) d\mu_N(w) \right] \right\} \]

showing the same structure as the rate functional (2.7). The product factor in Eq.(1.3) can be shown to induce terms of the subleading order \( O(\exp \{ N \}) \), and this is also valid for other contributions arising in the controlled derivation of LDT statement (2.6).
The event displacing the eigenvalue with the largest real part 
\( x_m = \text{Max}_i \{ x_i = \text{Re} z_i \} \) to any value to the left of the rightmost point \( \text{Re} z = 1 \) of the support of the equilibrium measure (2.4) necessarily implies shoving all the remaining \( N - 1 \) eigenvalues to the left as well, so that the corresponding measure \( \mu_N \neq \mu_{eq} \). Then the Eq.(2.6) implies that such displacement may only happen with an exponentially small probability of the order \( \text{Prob}(x_m < x) \sim \exp \{ -C(x)N^2 \} \), with some \( C(x) > 0 \) as long as \( x < 1 \). In contrast, the event of displacing the eigenvalue with the largest real part \( x_m \) to any value to the right of \( \text{Re} z = 1 \) can be done by simply splitting off that single eigenvalue from the rest, without effecting rest of the eigenvalues, so for large enough \( N \) we will still have \( \mu_N \approx \mu_{eq} \). We conclude that the knowledge of LDP for measures Eq.(2.6) is not enough to control such a situation.

Denote for any subset \( B \subseteq E_\mu \) by \( \chi(\mu_N \in B) \) the indicator function such that \( \chi(\mu_N \in B) = 1 \) if \( \mu_N \in B \) and \( \chi(\mu_N \in B) = 0 \) if \( \mu_N \notin B \). Thinking of the ensemble average as one performed over different realizations of the random measure \( \mu_N \) we have \( \text{Prob}(\mu_N \in B) = \langle \chi(\mu_N \in B) \rangle \). To this end, let us now denote by \( B_x \) the set of those measures in \( E_\mu \) such that the support of \( \mu \) lies in the half-plane \( \text{Re} z \leq x \). In this setup, the event that \( x_m \leq x \) is the same as the event that \( \mu_N \in B_x \), so that the LDT implies \( \text{Prob}(\mu_N \in B_x) \sim \exp \{ -C(x)N^2 \} \).

We will outline below how to establish the LDT for the right-most eigenvalue to be found beyond the right-edge of the 'equilibrium circle', which can be shown to have the form

\[
\text{Prob}(x_m > x) \approx e^{-NL(x)}, \quad x > 1
\]

with the rate function \( L(x) > 0 \) for \( x > 1 \) and minimized at \( L(x = 1) = 0 \). We thus see that such event is also exponentially penalized, however with the speed linear in \( N \).

To verify (2.8) we will consider the probability density \( \text{Prob}(x_m \in (x, x + dx)) = p_N(x) \, dx \) for any \( x > 1 \) and follow the ideas of \([25]\) where a similar question was addressed for Hermitian matrices with real eigenvalues. There exists an obvious relation between the probability density \( p_N(x) \) of the right-most eigenvalue position and the probability \( E_N(x) \) to have no eigenvalues with \( x_i = \text{Re} z_i > x \):

\[
p_N(x) = -\frac{dE_N}{dx}, \quad E_N(x) = \left\langle \prod_{i=1}^{N} \chi(x - x_i) \right\rangle
\]

where \( \chi(u) = 1 \) for \( u > 0 \) and \( \chi(u) = 0 \) otherwise and we omit the subscript 'GinOE' by angular brackets here and henceforth. As \( \chi(u) = 1 - \chi(-u) \) one can further expand

\[
E_N(x) = \left\langle \prod_{i=1}^{N} [1 - \chi(x_i - x)] \right\rangle
= 1 - \left\langle \sum_{i=1}^{N} \chi(x_i - x) \right\rangle + \left\langle \sum_{i \neq j} \chi(x_i - x)\chi(x_j - x) \right\rangle - \ldots
= 1 - \int_C R_1(z)\chi(\text{Re} z - x) \, d^2 z + \frac{1}{2!} \int_{C^2} R_2(z_1, z_2)\chi(\text{Re} z_1 - x)\chi(\text{Re} z_2 - x) \, d^2 z_1 \, d^2 z_2 + \ldots
\]

where we have used the definition of the correlation functions (1.7). Clearly, (2.1) or, equivalently, (2.4) imply that \( R_1(z) \) decays to zero as \( N \to \infty \) outside the support.
of the equilibrium density (we will characterize the rate of this decay quantitatively below) so that \( R_1(z) \chi(\text{Re}z - x) \ll 1 \) for \( x > 1 \). Using the ‘asymptotic independence’ (2.5) we therefore may conclude that all the terms which involve higher correlation functions with \( k > 1 \) are negligible for \( x > 1 \), and to the leading order approximation using \( z = x + iy \) we have

\[
(2.11) \quad p_N(x > 1) \approx \int_C R_1(z) \delta(\text{Re}z - x) \, d^2z = N \int_\mathbb{R} \langle \rho_N(x, y) \rangle \, dy,
\]

We see that the problem is reduced to finding the leading asymptotics of the mean density to the right of the equilibrium measure support, which for rescaled GinOE is at \( x = 1 \), but for the original GinOE happens at \( x = \sqrt{N} \). To this task we again consider (1.14) and (1.15) after rescaling \( x = \tilde{x} \sqrt{N} \), but now employ a more refined large-\( N \) asymptotics than (2.2):

\[
(2.12) \quad \frac{\Gamma(N - 1, Na)}{(N - 2)!} \approx \frac{1}{\sqrt{2\pi N}} \frac{1}{a(a - 1)} e^{-N[a - 1 - \ln a]}, \quad a > 1
\]

Using it one finds that keeping only the leading exponential terms

\[
(2.13) \quad \int_{-\infty}^\infty R_1^i(\tilde{x} \sqrt{N}, y) \, dy \sim e^{-2N \mathcal{L}(\tilde{x})}, \quad R_1^i(\tilde{x} \sqrt{N}) \sim e^{-N \mathcal{L}(\tilde{x})}, \quad \tilde{x} > 1
\]

with \( \mathcal{L}(\tilde{x}) = \frac{\tilde{x}^2 - 1}{2} - \ln x > 0 \), thus verifying (2.8) with \( L(x) = \mathcal{L}(\tilde{x}) \).

**Nonlinear generalization of May’s model**

Our goal is to attempt to go beyond the linearized May model (1.2). To this end we suggest to consider a system of \( N \) coupled non-linear autonomous ODEs in the form

\[
(2.14) \quad \frac{dx_i}{dt} = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N,
\]

where \( \mu > 0 \) and the components \( f_i(x) \) of the vector field \( f = (f_1, \ldots, f_N) \) are zero mean random functions of the state vector \( x = (x_1, \ldots, x_N) \) which we will assume to be smooth in every realization. We again will denote as \( \langle \cdots \rangle \) averaging over the associated distributions. To put this model in the context of the discussion above, if \( x_s \) is an equilibrium of (1.1), i.e., if \( -\mu x_e + f(x_s) = 0 \) then, in the immediate neighbourhood of \( x_s \) the system (1.1) reduces to May’s model (1.2) with \( y = x - x_s \) and \( J_{jk} = (\partial f_j/\partial x_k)|_{x_s} \).

Our next goal is to chose a generic nonlinear model which would be rich enough to allow description of May-Wigner instability as a feature of its global rather than local phase portrait, yet simple enough to allow analytical insights.

To visualise the global picture, it is helpful to consider first a special case of a gradient-descent flow, characterised by the existence of a potential function \( V(x) \) such that \( f = -\nabla V \). In this case, the system (1.1) can be rewritten as \( dx/dt = -\nabla L \), with \( L(x) = \mu|x|^2/2 + V(x) \) being the associated Lyapunov function describing the effective landscape. In the domain of \( L \), the state vector \( x(t) \) moves in the direction of the steepest descent, i.e., perpendicular to the level surfaces \( L(x) = h \) towards ever smaller values of \( h \). This provides a useful geometric intuition. The term \( \mu|x|^2/2 \) represents the globally confining parabolic potential, i.e., a deep well on the surface of \( L(x) \), which does not allow \( x \) to escape to infinity. At the same time the random potential \( V(x) \) may generate many local minima of \( L(x) \) (shallow wells) which will play the role of attractors for our dynamical system.
Moreover, if the confining term is strong enough then the full landscape will only be a small perturbation of the parabolic well, typically with a single stable equilibrium located close to $x = 0$. In the opposite case of relatively weak confining term, the disorder-dominated landscape will be characterised by a complicated random topology with many points of equilibria, both stable and unstable. Note that in physics, complicated energy landscapes is a generic feature of glassy systems with intriguingly slow long-time relaxation and non-equilibrium dynamics, see e.g. [26]. From that angle the properties of the random landscapes exemplified by the above Lyapunov function $L(x)$ and related models attracted considerable attention [27, 29, 30, 31].

The above picture of a gradient-descent flow is however only a very special case since the generic systems of ODEs (1.1) are not gradient. The latter point can easily be understood in the context of model ecosystems. For, by linearising a gradient flow in a vicinity of any equilibrium, one always obtains a symmetric community matrix, whilst the community matrices of model ecosystems are in general asymmetric. Note also a discussion of an interplay between non-gradient dynamics in random environment and glassy behaviour in [32].

To allow for a suitable level of generality we therefore suggest to choose the $N$-dimensional vector field $f(x)$ as a sum of ‘gradient’ and non-gradient (‘solenoidal’) contributions:

$$f_i(x) = -\frac{\partial V(x)}{\partial x_i} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\partial A_{ij}(x)}{\partial x_j}, \quad i = 1, \ldots, N,$$

where we require the matrix $A(x)$ to be antisymmetric: $A_{ij} = -A_{ji}$. Such a representation for fields $f: \mathbb{R}^3 \to \mathbb{R}^3$, smooth enough and vanishing at infinity sufficiently fast, into curl-free and divergence-free parts $f = \nabla V + \nabla \times A$, with $V: \mathbb{R}^3 \to \mathbb{R}$ and $A: \mathbb{R}^3 \to \mathbb{R}^3$ is the well-known Helmholtz decomposition. The meaning of this decomposition is that vector fields can be generically divided into a conservative irrotational component, sometimes called ‘longitudinal’, whose gradient connects the attractors or repellors and a solenoidal ‘incompressible’ curl field, also called ‘transversal’. Note that in the case of bounded domains with a boundary a nonzero ‘harmonic’ (i.e. simultaneously irrotational and incompressible) component may be present; however this component is always zero if the flow on the boundary is zero.

A natural generalization of the Helmholtz decomposition to higher dimensions is the so-called Hodge decomposition of differential forms.

**Theorem 2.1 (Hodge decomposition).** Let $\mathcal{M}$ be a compact, boundaryless, oriented Riemannian manifold, and $\Omega^k(\mathcal{M})$ be the space of smooth, weakly differentiable $k-$forms on $\mathcal{M}$, $d$ stands for the exterior derivative operator, and $\delta$ for the codifferential operator. Then any form $\omega \in \Omega^k(\mathcal{M})$ can be uniquely decomposed as

$$\omega = d\alpha + \delta\beta + \gamma,$$

where $\alpha \in \Omega^{k-1}(\mathcal{M})$, $\beta \in \Omega^{k+1}(\mathcal{M})$ and $\gamma \in \mathcal{H}^k(\mathcal{M})$ is harmonic, i.e. simultaneously $d\gamma = 0$ and $\delta\gamma = 0$. 

We are here interested in $k = 1$ since with any vector field $f = (f_1, \ldots, f_N)$ one can associate a differential 1-form as $\omega = \sum_i f_i dx^i$. Then the role of $\alpha$ will be played by a scalar function (0-form) $V(x)$ with $d\alpha = \sum_i \partial_i V dx^i$. The role of $\beta$ will be played by some 2-form which can always be written as $\beta = \sum_{i,k} A_{i,k} dx^i \wedge dx^k$, while $\gamma = \sum_i f_i dx^i$.
with some antisymmetric $A_{l,k}(x) = -A_{k,l}(x)$. To define the action of the codifferential operator $\delta$ one needs to specify the Riemannian metric $g_{ij}$ on $M$. We will use the $\mathbb{R}^N$ with the Euclidean metric $g_{ij} = \delta_{ij}^2$ for the role of $M$ in which case the action of the codifferential operator is especially simple: $\delta \beta = \sum_{i,j} \delta_j A_{ji} dx^i$.

We see that our choice (2.15) amounts to neglecting the harmonic component which can be made possible either by appropriately chosen boundary conditions at infinity, or simply restricting our consideration to the class of fields with no harmonic component.

Correspondingly, we will call $V(x)$ the scalar potential and the matrix $A(x)$ the vector potential. The normalising factor $1/\sqrt{N}$ in front of the sum on the right-hand side in (2.15) ensures that the transversal and longitudinal parts of $f(x)$ are of the same order of magnitude for large $N$.

Finally, to make the model as simple as possible and amenable to a rigorous and detailed mathematical analysis we choose the scalar potential $V(x)$ and the components $A_{ij}(x)$, $i < j$, of the vector potential to be statistically independent, zero mean Gaussian random fields, with smooth realisations and the additional assumptions of homogeneity (translational invariance) and isotropy reflected in the covariance structure:

\[
\langle V(x)V(y) \rangle = v^2 \Gamma_V(|x-y|^2)
\]

\[
\langle A_{ij}(x)A_{nm}(y) \rangle = a^2 T_A(|x-y|^2)(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) .
\]

Here the angular brackets $\langle ... \rangle$ stand for the ensemble average over all realisations of $V(x)$ and $A(x)$, and $\delta_{in}$ is the Kronecker delta: $\delta_{in} = 1$ if $i = n$ and zero otherwise.

For simplicity, we also assume that the functions $\Gamma_V(r)$ and $\Gamma_A(r)$ do not depend on $N$. This implies [33]

\[
\Gamma_\sigma(r) = \int_0^r \exp \left\{ -s r \right\} \gamma_\sigma(s) ds, \quad \sigma = A, V,
\]

where the ‘radial spectral’ densities $\gamma_\sigma(s) \geq 0$ have finite total mass: $\int_0^\infty \gamma_\sigma(s) ds < \infty$. We normalize these densities by requiring that $\Gamma_\sigma'(0) = \int_0^\infty s^2 \gamma_\sigma(s) ds = 1$. We assume the finiteness of second derivatives everywhere, which should be enough to ensure that our fields are smooth enough in any realization. The ratio

\[
\tau = \frac{v^2}{(v^2 + a^2)}, \quad 0 \leq \tau \leq 1,
\]

is a dimensionless measure of the relative strengths of the longitudinal and transversal components of $f(x)$: if $\tau = 0$ then $f(x)$ is divergence free and if $\tau = 1$ it is curl free. We also define the ‘May ratio’ of the relaxation rate to the characteristic value set by interaction as

\[
m = \mu/\mu_c, \quad \mu_c = \sqrt{N(a^2 + v^2)}
\]

The phase portrait of our system is controlled precisely by these two parameters: $m$ and $\tau$.

\[2\text{Though } \mathbb{R}^N \text{ is not compact, with due effort Thm. 2.1 can be extended to this case as well.}\]
Counting multiple equilibria via Kac-Rice formulas

The non-linear system (1.1) may have multiple equilibria whose number \( N_{\text{eq}} \) and locations \( x_1, \ldots, x_{N_{\text{eq}}} \) depend on the realisation of the random field \( f(x) \). For the above model with smooth realizations of random vector fields one may safely assume that above locations are isolated, that is one can find \( N_{\text{eq}} \) nonoverlapping balls such that each ball contains only a single equilibrium.

In this situation it seems natural to define the function

\[
P_{\text{eq}}(J) = \left\langle \frac{1}{N_{\text{eq}}} \sum_{k=1}^{N_{\text{eq}}} \prod_{i,j} \delta\left( J_{ij} + \mu \delta_{ij} - \frac{\partial f_k}{\partial x_j} \right) \right\rangle
\]

which may be interpreted as the probability density for the entries \( J_{ij} \) of the Jacobian sampled over all critical points. One of the central points of our approach is a possibility to rewrite the right-hand side using the Kac-Rice type identity valid for smooth enough functions \( G(x) \):

\[
\left\langle \sum_{k=1}^{N_{\text{eq}}} G (x_k) \right\rangle = \int_{\mathbb{R}^N} dx G(x) \delta \left( -\mu x + f(x) \right) |\det J(x)|
\]

where \( J(x) = -\mu \mathbf{1} + \frac{d}{dx} \), \( \delta(x) \) is the multivariate Dirac \( \delta \)-function and \( dx \) is the volume element in \( \mathbb{R}^N \). In this way we see that

\[
P_{\text{eq}}(J) = \left\langle \frac{1}{N_{\text{eq}}} \int_{\mathbb{R}^N} dx \delta(J - J(x)) \delta(-\mu x + f(x)) |\det J(x)| \right\rangle
\]

where the (random) number of equilibria is formally given by the integral

\[
N_{\text{eq}} = \int_{\mathbb{R}^N} \delta(-\mu x + f(x)) |\det J(x)| dx
\]

The number of stable equilibria \( N_{\text{st}} \) is given by a similar integral:

\[
N_{\text{st}} = \int_{\mathbb{R}^N} \delta(-\mu x + f(x)) |\det J(x)| \chi(\text{Re} J < 0) dx
\]

where the factor \( \chi(\text{Re} J < 0) \) ensures that all eigenvalues of the Jacobian matrix have negative real parts.

This, in particular, implies that the probability \( p_{\text{st}} \) for a given equilibrium to be stable is simply given by

\[
p_{\text{st}} = \int_{\mathbb{R}^{N \times N}} P_{\text{eq}}(J) \chi(\text{Re} J < 0) \ dJ = \left\langle \frac{N_{\text{st}}}{N_{\text{eq}}} \right\rangle
\]

It seems rather challenging to evaluate \( p_{\text{st}} \) due to correlations between the random denominator \( N_{\text{eq}} \) and the numerator \( N_{\text{st}} \). To get some insights in the problem we will be using as a proxy the annealed version \( p_{\text{st}}^{(a)} = \langle \frac{N_{\text{st}}}{N_{\text{eq}}} \rangle \). Our main goal
will be therefore to develop methods to evaluate \( \langle \mathcal{N}_{st} \rangle \) and \( \langle \mathcal{N}_{eq} \rangle \) for our model, with emphasis on the asymptotic analysis for \( N \gg 1 \). An important issue of how well \( p_{st}^{(a)} \) approximates \( p_{st} \) remains so far completely open.

The Kac-Rice formulas (3) yield the ensemble average of \( \mathcal{N}_{eq} \) and \( \mathcal{N}_{st} \) in terms of that of the modulus of the spectral determinant of the matrix \( (J_{ij})_{ij}^{N} \), \( J_{ij} = \partial f_i / \partial x_j \). We begin with

\[
\langle \mathcal{N}_{eq} \rangle = \int_{\mathbb{R}^N} \langle \delta(-\mu x + f(x)) \rangle \det \left( -\mu \delta_{ij} + J_{ij}(x) \right) \, dx,
\]

By our assumptions (2.16) – (2.17) the random field \( f(x) \) is homogeneous and isotropic. For such fields samples of \( f \) and \( J \) taken in one and the same spatial point \( x \) are uncorrelated, \( \langle f_i \cdot \partial f_j / \partial x_i \rangle = 0 \) for all \( i,j,l \). This is well known and can be checked by straightforward differentiation. In addition, the field \( f \) is Gaussian, hence the \( f(x) \) and \( J_{ij}(x) \) are actually statistically independent. This simplifies the evaluation of the integral in (3.7) considerably. Indeed, the statistical average in (3.7) factorizes and, and since due to stationarity \( \langle \det \left( -\mu \delta_{ij} + J_{ij}(x) \right) \rangle \) does not vary with \( x \). Introducing the Fourier-representation of the Dirac \( \delta \)-function we need only evaluate

\[
\langle \delta(-\mu x + f(x)) \rangle = \int_{\mathbb{R}^N} \frac{dk}{(2\pi)^N} e^{-i k \cdot x} \langle e^{i k \cdot f(x)} \rangle.
\]

Furthermore, at every spatial point \( x \) the vector \( f(x) \) is Gaussian with uncorrelated and identically distributed components,

\[
\langle f_i(x) f_j(x) \rangle = \delta_{ij} \sigma^2, \quad \sigma^2 = 2v^2 |\Gamma_v'(0)| + 2a^2 |\Gamma_A'(0)| \frac{N-1}{N}.
\]

Therefore \( \langle e^{i k \cdot f(x)} \rangle = e^{-\sigma^2 |k|^2/2} \), and evaluating the integral on the right-hand side in (3.8) one arrives at

\[
\langle \mathcal{N}_{eq} \rangle = \frac{1}{\mu^N} \langle \det \left( -\mu \delta_{ij} + J_{ij} \right) \rangle,
\]

thus bringing the original non-linear problem into the realms of the random matrix theory.

The probability (ensemble) distribution of the matrix \( J \) can easily be determined in closed form. Indeed, the matrix entries of \( J \) are zero mean Gaussian variables and their covariance structure, at spatial point \( x \), can be obtained from (2.16) – (2.17) by differentiation:

\[
\langle J_{ij} J_{nm} \rangle = \alpha^2 \left[ (1 + \epsilon_N) \delta_{in} \delta_{jm} + (\tau - \epsilon_N) (\delta_{jn} \delta_{im} + \delta_{ij} \delta_{mn}) \right],
\]

where \( \epsilon_N = (1 + \tau)/N \) and \( \alpha = 2 \sqrt{v^2 + a^2} \). Thus, to leading order in the limit \( N \to \infty \),

\[
J_{ij} = \alpha (X_{ij} + \sqrt{\tau} \delta_{ij} \xi),
\]

where \( X_{ij}, i,j = 1, \ldots, N \) are zero mean Gaussians with

\[
\langle X_{ij} X_{nm} \rangle = \delta_{in} \delta_{jm} + \tau \delta_{jm} \delta_{im},
\]

and \( \xi \) is a standard Gaussian, \( \xi \sim N(0,1) \), which is statistically independent of \( X = (X_{ij}) \). Note that for the divergence free fields \( f(x) \) (i.e., if \( \tau = 0 \)) the entries of \( J \) are statistically independent in the limit \( N \to \infty \), exactly as in May’s model. On the other side, if \( f(x) \) has a longitudinal component (\( \tau > 0 \)) then this implies positive correlation between the pairs of matrix entries of \( J \) symmetric about the
main diagonal: $\langle X_{ij} X_{ji} \rangle = \tau$ if $i \neq j$. Such distributions of the community matrix has also been used in the neighbourhood stability analysis of model ecosystems \cite{13}. Finally, in the limiting case of curl free fields ($\tau = 1$), the matrix $J$ is real symmetric.

The representation (3.10) comes in handy as it allows one to express (3.9) as a random matrix integral:

$$
\langle N_{\text{tot}} \rangle = \frac{N^N}{m^N} \int_\mathbb{R} \langle \det (x \, \delta_{ij} - X_{ij}) \rangle_{X_N} \frac{e^{-N\tau^2}}{\sqrt{2\pi/N}} \, dt,
$$

where $x = \sqrt{N}(m + t \sqrt{\tau})$ and the angle brackets $\langle \ldots \rangle_{X_N}$ stand for averaging over the real random $N \times N$ matrices $X$ defined in (3.11), see also (3.14). The same arguments yield also the ensemble average of the number of stable equilibria, $\langle N_{st} \rangle$ as

$$
\langle N_{st} \rangle = \frac{N^N}{m^N} \int_\mathbb{R} \langle \det (x \, \delta_{ij} - X_{ij}) | \chi_x(X) \rangle_{X_N} \frac{e^{-N\tau^2}}{\sqrt{2\pi/N}} \, dt,
$$

where $\chi_x(X) = 1$, if all $N$ eigenvalues of matrix $X$ have real parts less than the spectral parameter $x$, and $\chi_x(X) = 0$ otherwise.

It is obvious that we need to concentrate on the matrices $X_{ij}$. This one-parameter family of random matrices known as the ‘Real Elliptic Ensemble’ interpolates between the Gaussian Orthogonal Ensemble of real symmetric matrices (GOE, $\tau = 1$) and real Ginibre ensemble of fully asymmetric matrices (GinOE, $\tau = 0$) and have enjoyed considerable interest in the literature in recent years \cite{34, 35}. In the next section we give a very brief overview of this ensemble, emphasizing similarities with the Ginibre case studied by us in detail earlier.

Real Elliptic Ensemble: summary of main features

The joint probability density function $P_N(X)$ of the matrix entries in the elliptic ensemble of real Gaussian random matrices $X$ of size $N \times N$ is given by

$$
P_N(X) = Z_N(\tau)^{-1} \exp \left\{ -\frac{1}{2(1 - \tau^2)} \text{Tr} \left( X X^T - \tau X^2 \right) \right\},
$$

where $Z_N(\tau) = (2\pi)^{N^2/2} (1 - \tau)^{N(N - 1)/4} (1 + \tau)^{N(N + 1)/4}$ is the normalisation constant and $\tau \in [0, 1)$. It is straightforward to verify that the covariance of matrix entries $X_{ij}$ is given by the expression specified in (3.11).

The JPD of complex eigenvalues $z_i$, $i = 1, \ldots, N$ for every set $\mathcal{X}_{L,M}$ can be immediately written down by simple modifications of the Thm. (1.1) by noticing that $\text{Tr} (\tau X^2) = \sum_i z_i^2$ so that

$$
P^{(L,M)}_1(z_1, \ldots, z_N) = C_{L,M}(\tau)^{-1} \prod_{1 \leq j < k \leq N} |z_j - z_k| \times \exp \left\{ -\frac{1}{2(1 + \tau)} \sum_{1 \leq j \leq N} z_j^2 \right\} \prod_{1 \leq j \leq N} \text{erfc} \left( \sqrt{\frac{2}{1 - \tau^2}} |\text{Im} z_j| \right) 1_{\mathcal{X} \in \mathcal{X}_{L,M}},
$$

where $C_{L,M}(\tau)$ are the appropriate normalisation constants. Correspondingly, the Pfaffian structure described in Thms. (1.2) and (1.3) retain their validity up to
replacing the skew-symmetric measure $\mathcal{F}(z_1, z_2)$ with:

\begin{equation}
(3.16) \quad \mathcal{F}(z_1 x_1 + iy_1, z_2 = x_2 + iy_2) = e^{-\frac{i}{\tau} \frac{1}{2} (z_1 \bar{z}_1 + z_2 \bar{z}_2) / 2}
\times \left[ 2i \delta^2 (z_1 - \bar{z}_2) \text{sgn}(y_1) \text{erfc} \left( |y_1| \sqrt{\frac{2}{1 - \tau^2}} \right) + \delta(y_1) \delta(y_2) \text{sgn} (x_2 - x_1) \right].
\end{equation}

The kernel $\mathcal{K}_N(z_1, z_2)$ again can be found explicitly, in terms of which all the correlation functions can be readily represented. In this way, after some further work, one recovers the analogues of (1.14) and (1.15). Namely, denote

$$
\psi^{(\tau)}_k(z) = e^{-\frac{z^2}{2(1 + \tau)}} h^{(\tau)}_k(z)
$$

where $h^{(\tau)}_k(z)$ are rescaled Hermite polynomials of complex variable,

\begin{equation}
(3.17) \quad h^{(\tau)}_k(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-iz^2} \left( z \pm it \sqrt{2\tau} \right)^k dt = \frac{(\pm i \sqrt{N})^k}{\sqrt{\pi}} \sqrt{\frac{N}{2\tau}} \int_{\mathbb{R}} e^{-\frac{N}{\pi} (u^2 - \sqrt{\tau})^2} u^k du
\end{equation}

Then, for $z = x + iy$ we have

\begin{equation}
(3.18) \quad R^{(c)}_1(x, y) = 2i \text{sign}(y) \text{erfc} \left( \sqrt{\frac{2}{1 - \tau^2}} |y| \right) S^{(c)}_\tau(z, \bar{z})
\end{equation}

where

\begin{equation}
(3.19) \quad \mathcal{K}^{(c)}_N(z, \bar{z}) = \frac{1}{2(1 + \tau) \sqrt{2\pi}} \sum_{j=0}^{N-2} \psi^{(\tau)}_j(z) \bar{\psi}^{(\tau)}_j(z) \frac{\psi^{(\tau)}_{j+1}(z)}{j!}.
\end{equation}

whereas the density of real eigenvalues takes the form

\begin{equation}
(3.20) \quad R^{(r)}_1(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N-1} \frac{|\psi^{(\tau)}_k(x)|^2}{k!} + \frac{1}{\sqrt{2\pi} (1 + \tau) (N - 1)!} \psi^{(\tau)}_N(x) \int_0^x \psi^{(\tau)}_{N-1}(u) du.
\end{equation}

It is easy to check that for $\tau = 0$ the formulas (3.18) and (3.20) are reduced to (1.14) and (1.15), respectively.

Having in mind that we are mostly interested in extracting asymptotics for large $N \gg 1$, we rescale $x \rightarrow x\sqrt{N}$, $y \rightarrow y\sqrt{N}$ and by manipulating the integral representation for Hermite polynomials rewrite (3.19) for $z = x + iy$ as

\begin{equation}
(3.21) \quad \mathcal{K}^{(r)}_N(z, \bar{z}) = i \left( \frac{N}{2\pi} \right)^{3/2} \int_{\mathbb{R}} dp \int_{\mathbb{R}} dq \frac{\Gamma \left( N - 2, \frac{N}{\pi} (p^2 - q^2) \right)}{(N - 2)!} e^{-N [A_y(p) + B_z(q)]}
\end{equation}

where

\begin{equation}
(3.22) \quad A_y(p) = \frac{1 - \tau}{2\tau} p^2 + \frac{p\sqrt{2}}{\tau} y, \quad B_z(q) = \frac{1 + \tau}{2\tau} q^2 + \frac{iq\sqrt{2}}{\tau} x
\end{equation}

and we used the incomplete $\Gamma$–function defined in (1.13). This allows one to extract the leading $N \gg 1$ behaviour of the integrals in (3.21) by the saddle-point method, taking (2.2) into account, and using the large-$N$ asymptotics (2.3). In this way one quickly arrives to the limiting 'equilibrium' density of complex eigenvalues.
$\lim_{N \to \infty} \left( \rho_N^{(c)}(x,y) \right)$ known as the elliptic law, which we present below for the ‘rescaled’ ensemble $X/\sqrt{N}$:

$$\rho_N(z) \to \rho_{eq}(z) = \begin{cases} \frac{1}{\pi \sqrt{1 - \tau^2}}, & \frac{(\text{Re } z)^2}{1 + \tau^2} + \frac{(\text{Im } z)^2}{1 - \tau^2} \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

which replaces the ‘circular law’ (2.4) to the case of general $0 \leq \tau < 1$. This perfectly agrees with the numerical simulations, see the figure below.

Figure 1. Eigenvalues of Real Elliptic matrix of size $N = 100$ and variance $\alpha = 1$.

One further need to extend the LDT developed in [24] in order to describe how much the empirical measure $\mu_N(z)$ of the Elliptic Ensemble deviates from $\mu_{eq}(z)$ for large $N \gg 1$. Doing this one finds, see [17], that the probability $P_N(B) = \text{Prob}[\mu_N \in B]$ behaves again as predicted by (2.6) with the LDT rate functional given by

$$\mathcal{J}_r(\mu) = \frac{1}{2} \left[ \int_{\mathbb{C}} \left( \frac{x^2}{1 + \tau} + \frac{y^2}{1 - \tau} \right) d\mu(z) - \int_{\mathbb{C}^2} \log |z - w| d\mu(z) d\mu(w) \right] - 3/8$$

One can further show that the above functional is nonnegative: $\mathcal{J}[\mu] \geq 0$, and has a unique minimizer given by the equilibrium measure with the density uniform in the ellipse with half-axes $1 \pm \tau$, in full agreement with Eq.(3.23). Finally, analysing the asymptotic decay of the density of real and complex eigenvalues to the right of the ellipse, that is for $x > 1 + \tau$ in scaled variables, one proves the analogue of the LDT for the right-most eigenvalue of the Real Elliptic Ensemble to be found beyond the right-edge of the ’equilibrium ellipse’, which is again of the form analogous to (2.8):

$$\text{Prob}(x_m > x) \approx e^{-NL(x)}, \quad x > 1 + \tau$$
with the rate

\[ L(x) = \min\left\{ L^{(c)}(x), L^{(r)}(x) \right\}, \]

where

\[ L^{(r)}(x) = -\frac{1}{2} + \frac{x^2}{2(1 + \tau)} - \frac{1}{8\tau} (x - \sqrt{x^2 - 4\tau})^2 \ln \frac{x + \sqrt{x^2 - 4\tau}}{2}. \]

and \( L^{(c)}(x) = 2L^{(r)}(x) \) for any \( x > 1 + \tau \). Those two rates characterize the large deviation events of real and complex eigenvalues to be beyond the right edge of the equilibrium support, respectively, and one immediately sees that fluctuations of real eigenvalues far beyond the equilibrium support are more probable than that of the complex-conjugate pairs, as real eigenvalues have twice as smaller rate of decay, hence the large deviation tail will be controlled by these events.
LECTURE 4

Mean Number of Equilibria: asymptotic analysis
in the framework of Large Deviations

By shifting and rescaling the integration variables we can rewrite (3.12) for \( N_{\text{tot}} \) and (3.13) for \( N_{\text{st}} \) in the form most suitable for the asymptotic analysis which we for brevity present as

\[
\langle N_{\sigma} \rangle = \frac{1}{m^N} \int_{\mathbb{R}} D_N^{(\sigma)}(x) \frac{e^{-\frac{N(x-m)^2}{2}}}{\sqrt{2\pi N}} \, dx,
\]

where we introduced the label \( \sigma \) taking two values: \( \sigma = \text{st} \) or \( \sigma = \text{tot} \)

\[
D_N^{(\text{tot})}(x) = \langle |\det(x \delta_{ij} - Z_{ij})| \rangle_Z,
\]

\[
D_N^{(\text{st})}(x) = \langle |\det(x \delta_{ij} - Z_{ij})| \chi_x(Z) \rangle_Z
\]

where \( \langle \cdots \rangle_Z \) indicates expectation over the distribution of matrices \( Z = X/\sqrt{N} \) from the rescaled Real Elliptic Ensemble with JPD (cf. 3.14)

\[
P(Z) = C_N(\tau) e^{-\frac{N}{2(1-\tau^2)}}[\text{Tr} Z \tau^{-\tau} \text{Tr} Z^\tau], \quad \tau \in [0, 1]
\]

with the appropriate normalization constant \( C_N(\tau) \), and the indicator function \( \chi_x(Z) = 1 \), if all \( N \) complex eigenvalues \( z_i \) of the matrix \( Z \) have real parts \( Re z_i \) less than the real parameter \( x \), and \( \chi_x(Z) = 0 \) otherwise.

To be able to use the LDT in this setting we further rewrite Eqs.(4.3) for \( D_{N}^{(\sigma)}(x) \) in terms of the spectral measure \( \mu_N \) whose density is the empirical counting function \( \rho_N(z) = \sum_i \delta(z - z_i) \) as

\[
D_N^{(\text{tot})}(x) = \langle e^{N\Phi(x)} \rangle_Z, \quad D_N^{(\text{st})}(x) = \langle e^{N\Phi(x)} \chi_{\{\mu_N \in B_x\}} \rangle_Z.
\]

where we introduced the 'logarithmic potential' functional of the counting measure

\[
\Phi(x)(\mu) = \int_{\mathbb{C}} \ln|x - z| \, d\mu_N(z)
\]

and recall that \( B_x \) stands for the set of the measures supported in the complex \( z \)-plane to the left of \( \text{Re} z = x \) (and by \( B_x^c \) its complement). Now recall, very informally, a few very basic facts about the Large Deviation Theory.

Let \( N \) be an integer (eventually considered to be large \( N \gg 1 \)) and \( \alpha_N \to \infty \) be any sequence growing with \( N \) to infinity. Consider a random variable \( x_N \) indexed by \( N \), and taking values in some space \( E \), so that \( x_N \) is characterized by the probability \( P_N(x_N \in A) \) for \( x_N \) to be in any measurable subset \( A \subset E \). Then, in a nutshell, the distribution \( P_N \) satisfies the Large Deviation Principle (LDP) with the speed \( \alpha_N \) and the rate function \( 0 \leq I(x) < \infty \) if as long as \( N \to \infty \)

\[
P_N(x_N \in A) \approx \exp \{-\alpha_N \inf_A I(x)\}, \quad \forall A \subset E.
\]
Here the symbol $\approx$ describes the leading asymptotic behaviour, i.e. disregards both the pre-exponential multiplicative factors and subleading exponential terms which diverge with $N$ slower than $\alpha_N$. As the simplest example, let the random variable $x_N \in \mathbb{R}$ be real and characterized by the probability density $p_N(x)$ with respect to the Lebesgue measure. Then LDP is equivalent to writing $P_N(x_N \in [x, x + dx]) = p_N(x) dx \approx \exp\{-\alpha_N I(x)\} dx$.

One of the most important results of the Large Deviation Theory is known as the Laplace-Varadhan Theorem which we discuss now in the most basic instance. Namely, assume that the distribution of a real random variable $p$ satisfies the LDP with the speed $\alpha$ and the rate function $I(x)$, take any bounded real function $F(x_N)$ and consider the functional

\begin{equation}
Z_N(F) = \int_{\mathbb{R}} e^{-\tilde{\alpha}_N F(x)} dP_N := \left< e^{-\tilde{\alpha}_N F(x)} \right>_P
\end{equation}

where $\tilde{\alpha}_N \to \infty$ as $N \to \infty$, though in general $\tilde{\alpha}_N \neq \alpha_N$. The most simple version of the Laplace-Varadhan Theorem asserts that for $\tilde{\alpha}_N = \alpha_N$

\begin{equation}
\lim_{N \to \infty} \alpha^{-1}_N \log Z_N = -\inf_{E} (F + I)
\end{equation}

or equivalently $Z_N \approx e^{-\alpha_N \inf_{E} (F + I)}$ for $N \gg 1$. For the simplest case of a random real $x$ with the measure $P_N$ having a continuous density $p_N(x)$ this statement immediately follows from applying the Laplace steepest descent method to evaluation of the above integral, however the statement of the theorem is much stronger as it also holds true for random variables of much more general nature, like random measures. Note that the condition of boundedness of $F(x)$ can be further relaxed.

Note also that if $\tilde{\alpha}_N / \alpha_N \to 0$ as $N \to \infty$ asymptotically $Z_N \approx e^{-\tilde{\alpha}_N(N)I(x_\ast)}$ where $x_\ast$ is the global minimizer of the rate $I(x)$.

Using the last fact in our setting (recall the LDP (2.6) for the Elliptic Ensemble with the speed $\lambda_N = N^2 \gg N$) and ignoring for simplicity the fact that the ‘logarithmic potential’ functional defined in 4.6 is not trivially bounded due to logarithmic singularities (the step can be justified with due effort) immediately implies that

\begin{equation}
D_N^{(tot)}(x) = \left< e^{N\Phi(x)(\mu)} \right>_2 \approx e^{N\Phi(x)}, \quad \Phi(x) := \Phi(x)(\mu_{eq}) = \int_{\mathbb{C}} \ln|x - z| d\mu_{eq}(z),
\end{equation}

since the elliptic law measure $\mu_{eq}(z)$, see Eq.(3.23), is the global minimizer for the LDP rate function over all possible random measures. The integral in Eq.(4.10) can be performed explicitly, and yields (PROBLEM)

\begin{equation}
\Phi(x) = \begin{cases}
\frac{1}{8\pi} (|x| - \sqrt{x^2 - 4\tau})^2 + \ln \frac{|x| + \sqrt{x^2 - 4\tau}}{2}, & |x| > 1 + \tau \\
\frac{x^2}{2(1+\tau)} - \frac{1}{2}, & |x| < 1 + \tau
\end{cases}
\end{equation}

This allows to rewrite the expression (4.1) for $\langle N_{tot} \rangle$ with the required accuracy as

\begin{equation}
\langle N_{tot} \rangle = \frac{1}{mN} \int_{\mathbb{R}} e^{-N\mathcal{L}_{tot}(x)} \frac{dx}{\sqrt{2\pi\tau/N}}, \quad \mathcal{L}_{tot}(x) = \frac{(x - m)^2}{2\tau} - \Phi(x)
\end{equation}

and representing $\mathbb{R} = B_{-} \cup A \cup B_{+}$ with

$A = \{x < 1 + \tau\}, \quad B_{-} = \{x < -(1 + \tau)\}, \quad B_{+} = \{x > 1 + \tau\}$
write it further as the sum of three contributions: 

\[ \langle N_{tot} \rangle = \langle N^A_{tot} \rangle + \langle N^B_{tot} \rangle + \langle N^C_{tot} \rangle \]

where

\[ \langle N^A_{tot} \rangle = \frac{1}{mN} \int_A e^{-N L_{tot}(x)} \frac{dx}{\sqrt{2\pi \tau/N}} \]

\[ \langle N^B_{tot} \rangle = \frac{1}{mN} \int_{B_0} e^{-N L_{tot}(x)} \frac{dx}{\sqrt{2\pi \tau/N}}. \]

Now the integrals can be treated by the standard Laplace steepest descent method. For \( x \in A \) we have due to (4.11)

\[ L_{tot}(x) = \frac{(x - m)^2}{2\tau} - \frac{x^2}{2(1 + \tau)} + \frac{1}{2}, \quad \frac{d}{dx} L_{tot}(x) = \frac{x}{\tau(1 + \tau)} - \frac{m}{\tau}, \]

and since \( \frac{d^2 L_{tot}(x)}{dx^2} = \frac{1}{\tau(1 + \tau)} > 0 \) we see that for \( 0 \leq m < 1 \) the function \( L_{tot}(x) \) is minimized inside the interval \( x \in (0, 1 + \tau) \) at the point \( x_* = m(1 + \tau) \) where \( L_{tot}(x_*) = \frac{1}{2}(m^2 - 1) \). At the same time for \( m > 1 \) the function is minimized at the right end of the interval where \( L_{tot}(x = 1 + \tau) = \frac{(1 - m)^2}{2\tau} - (1 - m) \). This gives

\[ \langle N^A_{tot} \rangle \approx \begin{cases} e^{-N\left(\frac{1}{2}(m^2 - 1) + \ln m\right)}, & 0 \leq m < 1 \\ e^{-N\left(\frac{(1 - m)^2}{2\tau} - (1 - m) + \ln m\right)}, & m > 1 \end{cases} \]

Turning our attention to the integral for \( \langle N^B_{tot} \rangle \) we have due to (4.11) for \( x > 1 + \tau \):

\[ \frac{d}{dx} L_{tot}(x) = \frac{1}{\tau} \left( \frac{1}{2}(x + \sqrt{x^2 - 4\tau - m}) \right), \]

\[ \frac{d^2}{dx^2} L_{tot}(x) = \frac{1}{2\tau} \left( x + \sqrt{x^2 - 4\tau} - \frac{1}{4\tau} \right) \]

and look for local minimum satisfying \( \frac{d}{dx} L_{tot}(x) = 0 \). In the integration domain \( x > 1 + \tau \) we have \( x^2 > 4\tau + m \) and it is convenient to parametrize \( x = 2\sqrt{\tau} \cos \theta \) with \( \theta \geq 0 \) so that the equation for \( \theta \) takes the form \( m/\sqrt{\tau} = \exp\{\theta\} \). The solution \( \theta_* = \ln(m/\sqrt{\tau}) > 0 \) exists as long as \( m > \sqrt{\tau} \), implying finally that

\[ x_* = m + \frac{\tau}{m} \]

for which one finds

\[ \frac{d^2}{dx^2} L_{tot}(x_*) = \frac{m}{\tau} \left( \frac{1}{m - \frac{\tau}{m}} \right) > 0 \]

Therefore the function \( L_{tot}(x) \) has its global minimum at \( x = x_* \) as long as \( m > \sqrt{\tau} \). The condition \( x_* > 1 + \tau \) is equivalent to \( (m - 1) > (m - 1) \cdot (\tau/m) \).

If \( m > 1 \) the last condition implies \( m > \tau \) which is always satisfied as \( \tau < 1 \). Therefore for \( m > 1 \) the integral is dominated by the minimum, and one finds by direct calculations \( L_{tot}(x_* = -\ln m) \). In the case \( 0 \leq m < 1 \) the condition \( x_* > 1 + \tau \) implies \( m < \tau \) which is incompatible with \( m > \sqrt{\tau} \), so that the minimum is not operative any longer. In that case one finds \( \frac{d}{dx} L_{tot}(x) > 0 \) for any real \( x \) so the function increases monotonically and we conclude that the integral over the domain \( x \in (1 + \tau, \infty) \) for any \( 0 \leq m < 1 \) is dominated by the boundary
of integration \(x = 1 + \tau\) where \(L_{\text{tot}}(x = 1 + \tau) = \frac{(1-m)^2}{2\tau} - (1 - m)\). Combining everything

\[
\langle N_{\text{tot}}^B \rangle \approx \begin{cases} 
\exp\left(-N\left(\frac{(1-m)^2}{2\tau} -(1-m)+ \ln m\right)\right), & 0 \leq m < 1 \\
O(1), & m > 1
\end{cases}
\]

(4.19)

Comparing (4.19) to (4.14) we see that \(\langle N_{\text{tot}}^B \rangle \ll \langle N_{\text{tot}}^A \rangle\) for \(0 < m < 1\), whereas for \(m > 1\) the order is opposite: \(\langle N_{\text{tot}}^B \rangle \gg \langle N_{\text{tot}}^A \rangle\). Taking into account that for \(m > 0\) the contribution of \(\langle N_{\text{tot}}^B \rangle\) is negligible, we finally see that the mean number of equilibria satisfies

\[
\lim_{N \to \infty} \frac{1}{N} \ln \langle N_{\text{tot}} \rangle = \Sigma_{\text{tot}}(m)
\]

(4.20)

where \(\Sigma_{\text{tot}}(m) = \frac{1}{2}(m^2 - 1) - \ln m > 0\) for all \(0 < m < 1\) and \(\Sigma_{\text{tot}}(m) = 0\) for \(m > 1\). Therefore, if \(m < 1\) then \(\langle N_{\text{tot}} \rangle\) grows exponentially with \(N\) with the rate independent of the value of \(\tau\). Further analysis allowed one to find the leading sub-exponential factor in such growth which turned out to depend on \(\tau\) [16]. Moreover, the same analysis reveals that if \(m > 1\) then

\[
\lim_{N \to \infty} \langle N_{\text{tot}} \rangle = 1.
\]

(4.21)

showing that on average, and indeed with probability tending to one, the system possesses a single equilibrium which further can be shown to be stable.

One can also use the LDT methods to find the leading exponential asymptotics for \(D_N^\text{st}(\tau)\) which turns out to be given by

\[
D_N^\text{st}(\tau) \approx \begin{cases} 
\exp(N \Phi_1(x)), & x > 1 + \tau \\
\exp(-N^2 K_\tau(x) + N \Phi_2(x)), & x < 1 + \tau
\end{cases}
\]

(4.22)

where the function \(\Phi_1(x)\) defined for \(x > 1 + \tau\) coincides with \(\Phi(x)\) from the upper line of (4.11) whereas the explicit form of the functions \(K_\tau(x)\) and \(\Phi_2(x)\) is not actually needed for our purposes apart from the following properties:

1. The function \(K_\tau(x)\) defined for all \(x \leq 1 + \tau\) has its minimum at \(x = 1 + \tau\). At that point its value is zero: \(K_\tau(x = 1 + \tau) = 0\) and for any fixed \(0 \leq \tau < 1\) the behaviour close to the point of minimum is quadratic: \(K_\tau(x) = K''_\tau (x-1-\tau)^2 + O((x-1-\tau)^3)\) with some positive, \(\tau\)-dependent constant \(K''_\tau > 0\).
2. The function \(\Phi_1(x)\) defined for \(x > 1 + \tau\) and \(\Phi_2(x)\) defined for \(x < 1 + \tau\) satisfy a continuity property at the point \(x = 1 + \tau\):

\[
\lim_{x \to (1+\tau)+0} \Phi_2(x) = \lim_{x \to (1+\tau)-0} \Phi_1(x) = \frac{\tau}{2}
\]

(4.23)

where we used Eq.(4.11) and employed \(\pm 0\) to indicate the limit from the right/left, respectively.

The precise line of reasoning leading to (4.22) is somewhat long, and we refer to the paper [17] for the details of the procedure. To give an informal flavor of the ideas, we first notice that if \(x > 1 + \tau\) in a typical realization all eigenvalues will
be located to the left of the line \( \text{Re} \, z = x \), hence \( \chi(\mu_N \in B_x) = 1 \) with probability close to one, as in view of (3.25)

\[
\text{Prob}\{\mu_N \in B_x\} = \langle \chi(\mu_N \in B_x) \rangle = 1 - \text{Prob}(x_m > x) \approx 1 - e^{-NL(x)}, \quad x > 1 + \tau
\]

This immediately indicates why for \( x > 1 + \tau \) we should expect to have \( D_{N}^{(st)}(x) \approx D_{N}^{(tot)}(x) \) implying the first line of (4.22). At the same time, for \( x < 1 + \tau \) we have \( \chi(\mu_N \in B_x) = 1 \) with a small probability given by the Large Deviation Theory, cf. (2.6) and (4):

\[
(4.24) \quad \text{Prob}\{\mu_N \in B_x\} \approx e^{-N^2K_{\tau}(x)}, \quad K_{\tau}(x) = \inf_{\mu_N \in B_x} [J_{\tau}(\mu)] = J_{\tau}(\mu^*)
\]

where we denoted the corresponding minimizer under the above constraints (for a given values of \( x \) and \( \tau \)) as \( \mu^*(z) \). Finding the corresponding density \( \rho_{St}(z) \) explicitly is a highly nontrivial exercise in potential theory, which for the present case is only solved for the special case of purely gradient flow \( \tau = 1 \) [36]. Still, one can write

\[
(4.25) \quad D_{N}(x) \approx e^{N\Phi_2(x)} e^{-N^2K_{\tau}(x)}
\]

giving the top line of (4.22). Continuity is expected as for \( x \to 1 + \tau \) we expect \( \mu^* \to \mu_{eq} \) and \( \Phi_2(x) \to \Phi^{(2)}(\mu_{eq}) = \Phi_1(x) \).

Having Eqs. (4.22) at our disposal, it is a straightforward task to analyse exponential asymptotics of \( \langle N_{st} \rangle \) in (4.1). First, we again subdivide the integration domain \( \mathbb{R} \) into two parts: \( C_- = (-\infty, 1 + \tau) \) and \( C_+ = (1 + \tau, \infty) \) aiming to extract the large-\( N \) asymptotics of \( \langle N_{st} \rangle \) from

\[
(4.26) \quad \langle N_{st} \rangle = \langle N_{st}^{(-)} \rangle + \langle N_{st}^{(+)} \rangle,
\]

where, after denoting, symbolically, by \( A_N(x) \) and \( B_N(x) \) the implicit sub-exponential terms in the bottom and top lines of (4.22), we have

\[
(4.27) \quad \langle N_{st}^{(-)} \rangle = \frac{1}{m^N} \int_{C_-} B_N(x) e^{-N^2K_{\tau}(x)+N\mathcal{L}_{-}(x)} \frac{dx}{\sqrt{2\pi \tau/N}},
\]

with \( \mathcal{L}_+(x) = \frac{(x-m)^2}{2\tau} - \Phi_2(x) \) and

\[
(4.28) \quad \langle N_{st}^{(+)} \rangle = \frac{1}{m^N} \int_{C_+} A_N(x) e^{-N\mathcal{L}_{+}} \frac{dx}{\sqrt{2\pi \tau/N}},
\]

with \( \mathcal{L}_+(x) = \frac{(x-m)^2}{2\tau} - \Phi_1(x) \). Note that \( \langle N_{st}^{(+)} \rangle \) for \( N \gg 1 \) is obviously the same as \( \langle N_{st}^{B+} \rangle \) whose asymptotics is given by (4.19).

Turning our attention now to the integral featuring in Eq.(4.27) we immediately conclude that due to different scaling with \( N \) of terms in the exponential for large \( N \gg 1 \) the integral will be dominated by the vicinity of the boundary \( x = 1 + \tau \) where the function \( K_{\tau}(x) \) has its quadratic minimum. Evaluating the integral by Laplace method and using the \textit{continuity} property Eq.(4.23) we see that the leading exponential asymptotics of \( \langle N_{st}^{(-)} \rangle \) is the same as for \( \langle N_{st}^{(+)} \rangle \). This implies the mean number of stable equilibria given by:

\[
(4.29) \quad \lim_{N \to \infty} \frac{1}{N} \ln \langle N_{st} \rangle = \Sigma_{st}(m; \tau),
\]
here the 'complexity function' for any values of the control parameters \(0 < m < 1\) and \(0 < \tau \leq 1\) is given explicitly by

\[
\Sigma_{st}(m; \tau) = -\left[1 - m + \ln m + \frac{(1-m)^2}{2\tau}\right]
\]

Finally, as was already mentioned, the boundary case of purely gradient descent dynamics \(\tau = 1\) is equivalent to counting minima of certain random potentials, see discussion in [16]. That counting can be done by several methods, with or without the use of the large deviations techniques, see [28, 29, 30, 31], and the resulting complexity is exactly the same as given by Eq.(4.29) with \(\tau = 1\).

We can immediately infer that in the 'topologically non-trivial' regime of the \((m, \tau)\) parameter plane (see Fig. 1) there exists a curve \(\tau_B(m)\) given explicitly by

\[
\tau_B(m) = -\frac{1}{2} \frac{(1-m)^2}{1 - m + \ln m}, \quad 0 < m \leq 1, \quad 0 \leq \tau \leq 1
\]

such that for the parameter values below that line the 'complexity function' associated with the stable equilibria is negative: \(\Sigma_{st}(m; \tau) < 0\), implying that for such values of \(m\) and \(\tau\) the mean number of stable solutions is exponentially small. As the random variable \(N_{st}\) can take only non-negative integer values \(N_{st} = 0, 1, 2, \ldots\) this in turn implies that in a typical realization of random couplings \(f(x)\) in Eq.(1.1) there are simply no stable equilibria at all, i.e. \(N_{st} = 0\). Only in rare realizations, with exponentially small probability the variable \(N_{st}\) may take positive integer values. It is natural to name this type of the phase portrait as the 'absolute instability' regime. In contrast, for the parameter values above the curve \(\tau_B(m)\) the complexity function Eq.(4.30) is positive so that stable equilibria are exponentially abundant. Still, since \(\Sigma_{st}(m; \tau) < \Sigma_{tot}\) for any \(m < 1\), the stable equilibria are exponentially rare among all possible equilibria. One may call the associated type of the phase portrait as the 'relative instability' regime.

Discussion, Open Questions and further developments

In these lectures we attempted to explain the picture developed recently in [16] and [17] which suggests the following essential refinement of the May’s instability transition picture in the nonlinear setting. Let us fix the parameter \(\tau\) of the dynamical system Eq.(1.1) as defined in Eq.(2.18) to a value smaller than unity so that the corresponding dynamics is not of the pure gradient descent type and fix the value of the scaled relaxation rate \(m\) defined in Eq.(2.19) to \(m > 1\) to place the system initially in the topologically trivial regime with a single stable equilibrium. Then with decreasing the value of \(m\) our system will first experience an abrupt transition at \(m = m_C = 1\) from the topologically trivial stable regime to the 'absolute instability' regime extending for \(m_B < m < 1\), and then finally transits to the 'relative instability' regime for \(0 < m < m_B\), where the value \(m_B\) for a given \(\tau\) is given by the unique solution of the equation \(\tau_B(m) = \tau\), with the function \(\tau_B(m)\) defined in Eq.(4.31). In particular it is evident that \(m_B \to 1\) as \(\tau \to 1\) so that the 'absolute instability' regime is expected to be absent for the purely gradient descent flow. Actually, the case \(\tau = 1\) is equivalent to counting minima of certain random potentials, see discussion in [16]. That counting has been done earlier by several methods, with or without the use of the large deviations techniques, see [28, 29, 30, 31], and the resulting complexity is exactly the same as given by Eq.(4.29) and Eq.(4.30) with \(\tau = 1\). That case is clearly quite special, as not
only there is no regime of 'absolute instability', i.e. \( m_B = m_C \), but the complexity \( \Sigma_{st}(m;1) \) vanishes at the boundary of the corresponding 'relative instability' regime \( m = m_B = m_C = 1 \) cubically: \( \Sigma_{st}(m;1) \propto (1 - m)^3 \), whereas for any non-gradient dynamics it vanishes quadratically when approaching the value \( m = m_B \). Such a peculiarity is related to the 'third order' nature \([30]\) of the transition into the glassy phase for the associated random potential problem\([30]\).

For any \( \tau < 1 \) the 'absolute instability' regime does exist, and for weakly non-gradient system \( 1 - \tau \ll 1 \) its widths changes linearly: \( 1 - m_B \approx \frac{3}{2}(1 - \tau) \). In the opposite limit \( \tau \to 0 \) the value \( m_B \) tends to zero exponentially: \( m_B \sim \exp\left\{-\frac{1}{2\tau}\right\} \), so that for the purely solenoidal dynamics the 'relative instability' regime does not exist at all: there are no stable equilibria for any value of \( 0 < m < 1 \). All these features are evident from the figure Fig.1.

We expect this sharp transition in the phase portrait to be shared by other systems of randomly coupled autonomous ODE’s with large number of degrees of freedom, such as, e.g. a model of neural network consisting of randomly interconnected neural units \([4]\), or non-relaxational version of the spherical spin-glass model \([32]\). As was already mentioned, the model considered in \([4]\) is essentially of the form Eq.(1.1) but with the particular choice of \( f_i = \sum_j J_{ij} S(x_j) \) where \( S \) is an odd
sigmoid function representing the synaptic nonlinearity and $J_{ij}$ are independent centred Gaussian variables representing the synaptic connectivity between neuron $i$ and $j$. Although being Gaussian, the corresponding (non-gradient) vector field is not homogeneous (in particular, the origin $x = 0$ is always the point of an equilibrium) and thus seems rather different from our choice and not easily amenable to a rigorous analysis. Nevertheless, a recent paper [37] provided a strong argument about existence of a threshold in the the coupling strength such that beyond such a threshold the single equilibrium at $x = 0$ becomes unstable and exponentially many equilibria emerge instead for $x \neq 0$. In fact, in the vicinity of the threshold the complexity rate $\Sigma_{\text{tot}}$ estimated in that paper turned out to be given by exactly the same expression as in our model. Moreover, by combining the methods of the present paper with ideas of [37] one can infer that the complexity of stable equilibrium $\Sigma_{\text{st}}$ in the vicinity of the same threshold should be shared by the two models as well. This observation points towards considerable universality of the conclusions based on our model Eq.(1.1).

Our results should be certainly considered as only first steps towards deeper quantitative analysis of dynamical behaviour of generic complex systems of many randomly coupled degrees of freedom. Earlier studies, starting from the classical paper [4] suggested that for systems of such type dynamics in the ‘topologically nontrivial’ regime should be predominantly chaotic, see [37, 38] and references therein. The absence of stable, attracting equilibria certainly corroborates this conclusion, though presence of stable periodic orbits in the phase space can not be excluded on those grounds either. In particular, one may hope to be able to shed some light on the dynamical features by incorporating insights from recent research on statistics of the Lyapunov exponents in random flows [41]. The influence of the non-gradient (also termed ‘non-relaxational’) component of the vector field on systems dynamics needs further clarification as well. On one hand, as we discovered above any admixture of such components very efficiently eliminates all stable equilibria when entering the ‘topologically non-trivial’ regime. On the other hand, the results of the paper [32] suggest that the influence of non-relaxational components on long-time ‘aging’ effects in dynamics of glassy-type models is relatively benign. This may imply that the dynamical dominance of exponentially abundant, though unstable equilibria with yet extensively many both stable and unstable directions may be enough for ‘trapping’ the system dynamics for a long time in the vicinity of such equilibria thus inducing aging phenomena, similar to the gradient descent case addressed in [42]. Clarification of these intriguing questions remains a serious challenge. At the same time such issues as more detailed classification of unstable equilibria by the number of stable directions, addressing statistical characteristics of $N_{\text{tot}}$ and $N_{\text{st}}$ beyond their mean values, and finally investigating similar questions in other models with non-relaxational dynamics (see e.g. [43]) are within reach of the presently available methods and we hope to be able to address them in future publications.
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